

LOCALIZATION AND PROJECTIONS ON BI-PARAMETER BMO

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ABSTRACT. We prove that for any operator T on bi-parameter BMO the identity factors through T or $\text{Id} - T$. Bourgain's localization method provides the conceptual framework of our proof. It consists in replacing the factorization problem on the non-separable bi-parameter BMO by its localized, finite dimensional counterpart. We solve the resulting finite dimensional factorization problems by exploiting the geometry and combinatorics of colored dyadic rectangles.

1. INTRODUCTION

The dyadic intervals \mathcal{D} on the unit interval are given by

$$\mathcal{D} = \{[2^{-j}k, 2^{-j}(k+1)[: j, k \in \mathbb{N}_0, k \leq 2^j - 1\},$$

and the dyadic rectangles \mathcal{R} on the unit square by $\mathcal{R} = \mathcal{D} \times \mathcal{D}$. For any given dyadic interval $I \in \mathcal{D}$ we define the L^∞ normalized Haar function h_I , to be +1 on the left half of I and -1 on the right half of I . Given two dyadic intervals I, J we have

$$h_{I \times J}(s, t) = h_I(s) h_J(t), \quad s, t \in [0, 1[.$$

We define the bi-parameter space $H^1(\delta^2)$ to be the completion of

$$\text{span}\{h_{I \times J} : I \times J \in \mathcal{R}\}$$

under the norm

$$\|f\|_{H^1(\delta^2)} = \int_0^1 \int_0^1 \left(\sum a_{I \times J}^2 h_{I \times J}^2 \right)^{1/2} ds dt, \quad (1.1)$$

where f is the finite linear combination $f = \sum a_{I \times J} h_{I \times J}$. The dual of $H^1(\delta^2)$ is denoted $BMO(\delta^2)$. It consists of bi-parameter functions $f = \sum a_{I \times J} h_{I \times J}$ with $\|f\|_{BMO(\delta^2)} < \infty$, where

$$\|f\|_{BMO(\delta^2)} = \left(\sup_{\Omega \text{ open}} \frac{1}{|\Omega|} \sum_{I \times J \subset \Omega} a_{I \times J}^2 |I \times J| \right)^{1/2}. \quad (1.2)$$

For basic information and background we refer to [1], [3], [7], [9], [10], [12] and [14].

The main result of this paper is the following theorem.

Theorem 1.1 (Main Theorem). *For any operator*

$$S : BMO(\delta^2) \rightarrow BMO(\delta^2)$$

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the identity on $BMO(\delta^2)$ factors through $H = S$ or $H = \text{Id} - S$, that is

$$\begin{array}{ccc} BMO(\delta^2) & \xrightarrow{\text{Id}} & BMO(\delta^2) \\ E \downarrow & & \uparrow P \\ BMO(\delta^2) & \xrightarrow{H} & BMO(\delta^2) \end{array} \quad \|E\| \|P\| \leq C, \quad (1.3)$$

where $C > 0$ is some universal constant.

As a consequence, $BMO(\delta^2)$ is a primary Banach space. Recall that a Banach space X is primary if for any projection $S : X \rightarrow X$ one of the spaces $S(X)$ or $(\text{Id}_X - S)(X)$ is isomorphic to X . For background on this classical isomorphic invariant concept we refer to [13, 17, 22].

One cannot directly deduce the result that $BMO(\delta^2)$ is primary from the previously known result that $H^1(\delta^2)$ is primary [16]. To see this, we remark that there exists a projection on $BMO(\delta^2)$ that is not weak* continuous, and therefore is not the adjoint of an operator on the predual $H^1(\delta^2)$. Indeed, given a Banach limit $\ell : \ell^\infty \rightarrow \mathbb{R}$ and a collection of disjoint dyadic rectangles $\{R_1, R_2, \dots\}$, let us define the rank one projection $S : BMO(\delta^2) \rightarrow BMO(\delta^2)$ by

$$Sf = \ell\left(\left(\frac{\langle f, h_{R_n} \rangle}{|R_n|}\right)_{n=1}^\infty\right) \sum_{j=1}^\infty h_{R_j}.$$

One can easily verify that S is a bounded linear projection that is not weak* continuous, since $f_n = \sum_{j \geq n} h_{R_j}$ is a weak* null sequence and $Sf_n = \sum_{j=1}^\infty h_{R_j}$.

Our proof of the main theorem is based on the localization method introduced by J. Bourgain in [4]. See also [5, 6] and [19] for one of the first papers in this direction. Bourgain's method is particularly useful for treating factorization problems on non-separable Banach spaces such as $BMO(\delta^2)$. It aims at replacing (1.3) by its localized, finite dimensional counterpart, and in our context it consists of three basic steps.

- (i) The starting point consists in applying Wojtaszczyk's isomorphism [21] to the space $BMO(\delta^2)$ and its finite dimensional building blocks $BMO_n(\delta^2) = \text{span}\{h_{I \times J} : I \times J \in \mathcal{R}, |I|, |J| \geq 2^{-n}\} \cap BMO(\delta^2)$. This gives

$$BMO(\delta^2) \sim \left(\sum_n BMO_n(\delta^2) \right)_\infty,$$

where we use the notation $X \sim Y$ to denote that X is isomorphic to Y .

- (ii) Reduction to diagonal operators on $\left(\sum_n BMO_n(\delta^2) \right)_\infty$.
- (iii) Verification of the following finite dimensional and quantitative factorization problem: For any $n \in \mathbb{N}$ and $M > 0$ there exists $N = N(n, M)$ such that for any operator $T : BMO_N(\delta^2) \rightarrow BMO_N(\delta^2)$ with $\|T\| \leq M$ we have that $H = T$ or $H = \text{Id} - T$ satisfies

$$\begin{array}{ccc} BMO_n(\delta^2) & \xrightarrow{\text{Id}} & BMO_n(\delta^2) \\ E \downarrow & & \uparrow P \\ BMO_N(\delta^2) & \xrightarrow{H} & BMO_N(\delta^2) \end{array} \quad \|E\| \|P\| \leq C, \quad (1.4)$$

where $C > 0$ is some universal constant.

The most challenging aspect in connection with the localization method of Bourgain consists in proving the finite dimensional factorization problem (1.4) while simultaneously controlling N in terms of n . The one-parameter factorization problems

– solved in [15] – are both the model case and also a special case of our present problem. See also [2, 17, 18, 20].

Organization of the paper.

Section 2 lists the preliminary theorems, definitions and concepts. Section 3 states our main technical results (finite dimensional quantitative factorization and almost-diagonalization theorems). Section 4 contains the proof of the almost-diagonalization theorem. Section 5 restates the main theorem (infinite dimensional factorization) and gives its detailed proof.

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2. PRELIMINARIES

Basic notation.

Here we collect basic notation and definitions. We refer to [17] for reference. Recall that \mathcal{D} denotes the dyadic subintervals of the unit interval. Let $\pi : \mathcal{D} \setminus \{[0, 1]\} \rightarrow \mathcal{D}$ denote the dyadic predecessor map, that is $\pi(I) = \bigcap \{J \in \mathcal{D} : J \supsetneq I\}$. The level $\text{lev}(I)$ of a dyadic interval $I \in \mathcal{D}$ is defined as $\text{lev}(I) = -\log_2(|I|)$. The collection \mathcal{D}_j of dyadic intervals at level j is given by $\mathcal{D}_j = \{I \in \mathcal{D} : \text{lev}(I) = j\}$ and we set $\mathcal{D}^n = \bigcup_{j \leq n} \mathcal{D}_j$. For $n \in \mathbb{N}$ we define

$$\mathcal{R}_n = \{I \times J \in \mathcal{R} : I, J \in \mathcal{D}^n\}.$$

Given a collection of sets \mathcal{C} we define

$$\mathcal{C}^* = \bigcup \{C : C \in \mathcal{C}\}.$$

If A is some set, then $\mathcal{C} \cap A = \{C \cap A : C \in \mathcal{C}\}$. The Carleson constant $[\![\mathcal{A}]\!]$ of a collection $\mathcal{A} \subset \mathcal{D}$ is given by

$$[\![\mathcal{A}]\!] = \sup_{I \in \mathcal{A}} \sum_{J \in \mathcal{A} \cap I} |J|/|I|.$$

Note that $[\![\mathcal{A} \cup \mathcal{B}]\!] \leq [\![\mathcal{A}]\!] + [\![\mathcal{B}]\!]$ for any two collections $\mathcal{A}, \mathcal{B} \subset \mathcal{D}$.

For any given dyadic interval $I \in \mathcal{D}$ we define $h_I = \mathbb{1}_{I_0} - \mathbb{1}_{I_1}$, where $\mathbb{1}_A$ denotes the characteristic function of a set A , $I_0 = [\inf I, (\inf I + \sup I)/2[$ and $I_1 = [(\inf I + \sup I)/2, \sup I[$. The one parameter hardy space $H^1(\delta)$ is the completion of

$$\text{span}\{h_I : I \in \mathcal{D}\}$$

under the square function norm

$$\|f\|_{H^1(\delta)} = \int_0^1 \left(\sum a_I^2 h_I^2 \right)^{1/2} dt,$$

where $f = \sum a_I h_I$. We set

$$\begin{aligned} H_n^1(\delta) &= \text{span}\{h_I : I \in \mathcal{D}^n\} \cap H^1(\delta), \\ BMO_n(\delta) &= \text{span}\{h_I : I \in \mathcal{D}^n\} \cap BMO(\delta). \end{aligned}$$

The Gamlen–Gaudet factorization.

We recall the relation between large Carleson constants and factorization, see [17]. Let \mathcal{A} be a collection of dyadic intervals satisfying $[\![\mathcal{A}]\!] \geq N$ and define

$$X_{\mathcal{A}} = \text{span}\{h_I : I \in \mathcal{A}\} \cap H^1(\delta).$$

If $N = n4^n$, then there exist linear operators E and P so that

$$\begin{array}{ccc} H_n^1(\delta) & \xrightarrow{\text{Id}} & H_n^1(\delta) \\ & \searrow E & \nearrow P \\ & X_{\mathcal{A}} & \end{array} \quad \|E\|\|P\| \leq C. \quad (2.1)$$

The bi-parameter analogues of these finite dimensional building blocks are:

$$\begin{aligned} H_n^1(\delta^2) &= \text{span}\{h_{I \times J} : I \times J \in \mathcal{R}_n\} \cap H^1(\delta^2), \\ BMO_n(\delta^2) &= \text{span}\{h_{I \times J} : I \times J \in \mathcal{R}_n\} \cap BMO(\delta^2). \end{aligned}$$

In the bi-parameter context, factorization and large Carleson constants are related for collections of rectangles having product structure. Given collections of dyadic intervals $\mathcal{A}, \mathcal{B} \subset \mathcal{D}$ we define the product space $X_{\mathcal{A} \times \mathcal{B}}$ by

$$X_{\mathcal{A} \times \mathcal{B}} = \text{span}\{h_{I \times J} : I \times J \in \mathcal{A} \times \mathcal{B}\} \cap H^1(\delta^2).$$

If $N = n4^n$ and $[\![\mathcal{A}]\!] \geq N$, $[\![\mathcal{B}]\!] \geq N$, then there exist linear operators E and P such that

$$\begin{array}{ccc} H_n^1(\delta^2) & \xrightarrow{\text{Id}} & H_n^1(\delta^2) \\ & \searrow E & \nearrow P \\ & X_{\mathcal{A} \times \mathcal{B}} & \end{array} \quad \|E\|\|P\| \leq C. \quad (2.2)$$

Due to product structure of $X_{\mathcal{A} \times \mathcal{B}}$, the bi-parameter factorization (2.2) results directly from its one-parameter predecessor (2.1). In the next paragraph we discuss Ramsey's theorem for colored dyadic rectangles. Its relevance for the constructions of this paper comes from the fact that for any two-coloring of \mathcal{R}_n , Ramsey's theorem detects a large monochromatic collection of the form $\mathcal{A} \times \mathcal{B}$.

Ramsey theorem for colored dyadic rectangles.

Ramsey's theorem asserts that for any two-coloring of the dyadic rectangles

$$\mathcal{R}_n = \{I \times J : |I| \geq 2^{-n}, |J| \geq 2^{-n}\}$$

there exist collections \mathcal{A}, \mathcal{B} of dyadic intervals, each of which has large Carleson constant and, moreover,

$$\mathcal{A} \times \mathcal{B} \text{ is monochromatic in } \mathcal{R}_n.$$

For later reference we state this assertion in the following theorem.

Theorem 2.1. *Given $n_0 \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for any collection $\mathcal{C} \subset \mathcal{R}_n$ one finds $\mathcal{A}, \mathcal{B} \subset \mathcal{D}$ satisfying*

- (i) $\mathcal{A} \times \mathcal{B} \subset \mathcal{C}$ or $\mathcal{A} \times \mathcal{B} \subset \mathcal{R}_n \setminus \mathcal{C}$,
- (ii) $[\![\mathcal{A}]\!] \geq n_0$ and $[\![\mathcal{B}]\!] \geq n_0$.

One can choose $n = n_0 2^{4^{n_0}}$.

For the above formulation of Ramsey's theorem we refer to [16]. See also [11, Chapter 1]. For convenience, we give the proof here.

Proof. Define $n = n_0 2^{4^{n_0}}$ and let $\mathcal{C} \subset \mathcal{R}_n$. Define $k = 2n_0 - 1$ and let $I_1, \dots, I_{2^{k+1}-1}$ be an enumeration of the dyadic intervals in \mathcal{D}^k . First, we set $\mathcal{E}_0 = \mathcal{F}_0 = \mathcal{G}_0 = \mathcal{D}^n$, $I_0 = \emptyset$ and $f(I_0) = \infty$. Second, assuming that $\mathcal{E}_j, \mathcal{F}_j, \mathcal{G}_j$ and $f(I_j)$ have already been constructed for all $0 \leq j \leq m-1$, we define the collections

$$\mathcal{E}_m = \{J \in \mathcal{G}_{m-1} : I_m \times J \notin \mathcal{C}\} \quad \text{and} \quad \mathcal{F}_m = \{J \in \mathcal{G}_{m-1} : I_m \times J \in \mathcal{C}\}.$$

If $\llbracket \mathcal{E}_m \rrbracket \geq \llbracket \mathcal{F}_m \rrbracket$, we set $f(I_m) = 0$, and if $\llbracket \mathcal{E}_m \rrbracket < \llbracket \mathcal{F}_m \rrbracket$, we set $f(I_m) = 1$. To conclude the inductive step, we define $\mathcal{G}_m = \mathcal{E}_m$ if $f(I_m) = 0$, and $\mathcal{G}_m = \mathcal{F}_m$ if $f(I_m) = 1$.

Observe that $\mathcal{E}_m \cup \mathcal{F}_m = \mathcal{G}_{m-1}$ and $\llbracket \mathcal{G}_m \rrbracket = \max(\llbracket \mathcal{E}_m \rrbracket, \llbracket \mathcal{F}_m \rrbracket)$. By subadditivity of $\llbracket \cdot \rrbracket$ we obtain $2\llbracket \mathcal{G}_m \rrbracket \geq \llbracket \mathcal{G}_{m-1} \rrbracket$, and by iterating $\llbracket \mathcal{G}_m \rrbracket \geq 2^{-m}(n+1)$. For $m = 2^{k+1} - 1$, we set $\mathcal{B} = \mathcal{G}_m$, and observe that by our choices $\llbracket \mathcal{B} \rrbracket \geq 2^{-4^{n_0}+1}(n+1) \geq n_0$. Now define

$$\mathcal{H}_0 = \{I \in \mathcal{D}^k : f(I) = 0\} \quad \text{and} \quad \mathcal{H}_1 = \{I \in \mathcal{D}^k : f(I) = 1\},$$

and note that $\mathcal{H}_0 \cup \mathcal{H}_1 = \mathcal{D}^k$. By definition of f and \mathcal{G}_j we have

$$\mathcal{H}_0 \times \mathcal{B} \subset (\mathcal{D}^k \times \mathcal{D}^k) \setminus \mathcal{C} \quad \text{and} \quad \mathcal{H}_1 \times \mathcal{B} \subset \mathcal{C}.$$

Now, let $\mathcal{A} = \mathcal{H}_0$ if $\llbracket \mathcal{H}_0 \rrbracket \geq \llbracket \mathcal{H}_1 \rrbracket$, and $\mathcal{A} = \mathcal{H}_1$ if $\llbracket \mathcal{H}_0 \rrbracket < \llbracket \mathcal{H}_1 \rrbracket$. We conclude this proof by observing that by our choices we have

$$2\llbracket \mathcal{A} \rrbracket \geq \llbracket \mathcal{H}_0 \rrbracket + \llbracket \mathcal{H}_1 \rrbracket \geq \llbracket \mathcal{H}_0 \cup \mathcal{H}_1 \rrbracket = \llbracket \mathcal{D}^k \rrbracket = k + 1 = 2n_0. \quad \square$$

Block bases and projections in $H^1(\delta^2)$.

We introduce next some frequently used terminology and record a boundedness criterion for projections on $H^1(\delta^2)$. We say that a sequence $\{b_{I \times J} : I \times J \in \mathcal{R}\}$ in a Banach space X is equivalent to the unconditional 2D Haar basis $\{h_{I \times J} : I \times J \in \mathcal{R}\}$ in $H^1(\delta^2)$ if the following holds: The map

$$T : \sum a_{I \times J} h_{I \times J} \rightarrow \sum a_{I \times J} b_{I \times J}$$

defined initially on finite linear combinations of 2D Haar functions and extended by density to $H^1(\delta^2)$ satisfies

$$C_1^{-1} \|x\|_{H^1(\delta^2)} \leq \|T(x)\|_X \leq C_1 \|x\|_{H^1(\delta^2)}, \quad x \in H^1(\delta^2).$$

Let $\{\mathcal{E}_{I \times J} : I \times J \in \mathcal{R}\}$ be pairwise disjoint collections of dyadic rectangles and let $E_{I \times J} = \mathcal{E}_{I \times J}^* = \bigcup_{K \times L \in \mathcal{E}_{I \times J}} K \times L$ be the point-set covered by the collection $\mathcal{E}_{I \times J}$.

We denote by

$$b_{I \times J} = \sum_{K \times L \in \mathcal{E}_{I \times J}} h_{K \times L}$$

the block-basis generated by $\mathcal{E}_{I \times J}$. We assume throughout, that $\|b_{I \times J}\|_2^2 = |E_{I \times J}|$ or equivalently that $\mathcal{E}_{I \times J}$ consists of pairwise disjoint dyadic rectangles. We formulate conditions on the collections $\{\mathcal{E}_{I \times J}\}$ so that the block basis $\{b_{I \times J}\}$ is equivalent to the 2D Haar system. The sets $\{E_{I \times J} : I \times J \in \mathcal{R}\}$ satisfy the **bi-tree condition** if there exists $C_2 > 0$ so that for each $I \times J \in \mathcal{R}$

$$C_2^{-1} |I \times J| \leq |E_{I \times J}| \leq C_2 |I \times J|. \quad (2.3a)$$

and for $(I_0 \times J_0), (I_1 \times J_1) \in \mathcal{R}$ with $I = \tilde{I}_0 = \tilde{I}_1$ and $J = \tilde{J}_0 = \tilde{J}_1$ we have

$$E_{I_0 \times J} \cap E_{I_1 \times J} = \emptyset, \quad E_{I_0 \times J} \cup E_{I_1 \times J} \subset E_{I \times J}, \quad (2.3b)$$

$$E_{I \times J_0} \cap E_{I \times J_1} = \emptyset, \quad E_{I \times J_0} \cup E_{I \times J_1} \subset E_{I \times J}. \quad (2.3c)$$

If (2.3) is satisfied, then the block basis $\{b_{I \times J} : I \times J \in \mathcal{R}\}$ is equivalent to the 2D Haar system in $H^1(\delta^2)$ and $C_1 = C_1(C_2)$. The following theorem is a basic tool that allows to project onto the span of the block bases $\{b_{I \times J} : I \times J \in \mathcal{R}\}$. It

was instrumental in proving that $H^1(\delta^2)$ is a primary space, see [8] and [16]. In the present paper, the main component of the factoring operator P appearing in Theorem 1.1 consists of a weighted version of the following orthogonal projection Q onto block basis.

Theorem 2.2. *Let $\mathcal{E}_{I \times J}$, $I \times J \in \mathcal{R}$ be pairwise disjoint collections consisting of disjoint dyadic rectangles. Let $E_{I \times J} = \mathcal{E}_{I \times J}^*$. Assume that $\{E_{I \times J} : I, J \in \mathcal{D}\}$ is a bi-tree, then the following hold*

- (i) *The block basis $\{b_{I \times J} : I \times J \in \mathcal{R}\}$ is equivalent to the 2D-Haar basis in $H^1(\delta^2)$ with $C_1 = C_1(C_2)$.*
- (ii) *If there exists $C_3 > 0$ so that for each $I \times J, I_0 \times J_0 \in \mathcal{R}$ with $I \times J \supset I_0 \times J_0$ and for every $K \times L \in \mathcal{E}_{I \times J}$ we have*

$$C_3^{-1} \frac{|E_{I_0 \times J_0}|}{|E_{I \times J}|} \leq \frac{|(K \times L) \cap E_{I_0 \times J_0}|}{|K \times L|} \leq C_3 \frac{|E_{I_0 \times J_0}|}{|E_{I \times J}|}, \quad (2.4)$$

then the orthogonal projection

$$Qf = \sum \langle f, b_{I \times J} \rangle \frac{b_{I \times J}}{\|b_{I \times J}\|_2^2}$$

defines a bounded operator on $H^1(\delta^2)$ with norm only depending on C_3 and C_2 .

Rademacher type functions in $H^1(\delta^2)$ and $BMO(\delta^2)$.

We define the following Rademacher type system as block basis of the Haar system. Given $r \geq k_0$ and $K_0 \times L_0 \in \mathcal{R}$ with $|K_0| = 2^{-k_0}$ we specify the following functions. First, for any choice of signs we set

$$d_i = \sum_{K \in D_i \cap K_0} \pm h_K, \quad i \geq r.$$

Then it is easy to see that if we define

$$g_i(s, t) = d_i(s) h_{L_0}(t), \quad s, t \in [0, 1]$$

for each dyadic interval L_0 , then by (1.1) and duality we have

$$\left\| \sum_{i=r}^{r+k-1} g_i \right\|_{H^1(\delta^2)} = \sqrt{k} |L_0| \quad \text{and} \quad \left\| \sum_{i=r}^{r+k-1} g_i \right\|_{BMO(\delta)} = \sqrt{k}. \quad (2.5)$$

3. LOCALIZED FACORIZATION

Here we prove our quantitative factorization theorem which is one of the three major steps towards the proof of our main theorem.

The main result of this paper is the following quantitative factorization theorem 3.1.

Theorem 3.1. *For $n \in \mathbb{N}$ and $M > 0$ there exists $N = N(n, M)$ so that the following holds: For any operator $T : H_N^1(\delta^2) \rightarrow H_N^1(\delta^2)$ with $\|T\| \leq M$ the identity on $H_n^1(\delta^2)$ factors through $H = T$ or $H = \text{Id} - T$ such that*

$$\begin{array}{ccc} H_n^1(\delta^2) & \xrightarrow{\text{Id}} & H_n^1(\delta^2) \\ E \downarrow & & \uparrow P \\ H_N^1(\delta^2) & \xrightarrow{H} & H_N^1(\delta^2) \end{array} \quad \|E\| \|P\| \leq C,$$

where $C > 0$ is a universal constant.

The proof is based on the following three theorems.

- (i) Ramsey's theorem 2.1 for colored dyadic rectangles.
- (ii) The projection theorem 2.2.
- (iii) The almost-diagonalization theorem 3.2 stated below.

These three theorems combined provide the reduction from general operators in theorem 3.1 to multipliers on the Haar system.

The almost-diagonalization theorem.

We now state the almost-diagonalization theorem 3.2 and show that in combination with Ramsey's theorem 2.1 for colored dyadic rectangles and the projection theorem 2.2 it yields the proof of our main result, Theorem 3.1.

Theorem 3.2. *Let $n \in \mathbb{N}$, $M > 0$ and $\{\varepsilon_{I \times J} : I \times J \in \mathcal{R}_n\}$ be a given set of small positive scalars. Then there exists $N = N(n, M, \{\varepsilon_{I \times J}\})$ such that for any linear operator $T : H_N^1(\delta^2) \rightarrow H_N^1(\delta^2)$ with $\|T\| \leq M$ there exist disjoint collections $\mathcal{E}_{I \times J}$, indexed by $I \times J \in \mathcal{R}_n$, consisting of pairwise disjoint dyadic rectangles defining the functions*

$$b_{I \times J} = \sum_{K \times L \in \mathcal{E}_{I \times J}} h_{K \times L},$$

which satisfy the following conditions:

- (i) $\mathcal{E}_{I \times J} \subset \mathcal{R}_N$ and $|b_{I \times J}| \leq 1$ for all $I \times J \in \mathcal{R}_n$.
- (ii) The orthogonal projection

$$Q(f) = \sum_{I \times J \in \mathcal{R}_n} \left\langle f, \frac{b_{I \times J}}{\|b_{I \times J}\|_2} \right\rangle \frac{b_{I \times J}}{\|b_{I \times J}\|_2}$$

is a bounded operator on $H^1(\delta^2)$ with $Q(H^1(\delta^2)) = \text{span}\{b_{I \times J}\}$ satisfying

$$\|Q : H^1(\delta^2) \rightarrow H^1(\delta^2)\| \leq C_2,$$

for some universal constant $C_2 > 0$.

- (iii) The map $S : H_n^1(\delta^2) \rightarrow \text{span}\{b_{I \times J}\} \cap H_N^1(\delta^2)$ defined as the linear extension of $h_{I \times J} \mapsto b_{I \times J}$ is an isomorphism with

$$\|S\| \|S^{-1}\| \leq C_3, \tag{3.1}$$

for some universal constant $C_3 > 0$.

- (iv) We have the estimate

$$\sum_{K \times L \neq I \times J} |\langle T b_{K \times L}, b_{I \times J} \rangle| \leq \varepsilon_{I \times J} \|b_{I \times J}\|_2^2, \tag{3.2}$$

for all $I \times J \in \mathcal{R}_n$.

The proof of the almost-diagonalization theorem 3.2 is given in Section 4.

Proof of Theorem 3.1.

Let $n \in \mathbb{N}$, $M > 0$. We define N by the chain of the following conditions:

$$N = N(N_1, M, \{\varepsilon_{I \times J}\}), \quad N_1 = N_2 2^{4^{N_2}}, \quad N_2 = n 4^n, \tag{3.3}$$

where $\{\varepsilon_{I \times J} : I \times J \in \mathcal{R}_{N_1}\}$ is a collection of positive scalars satisfying

$$\sum_{I \times J \in \mathcal{R}_{N_1}} \varepsilon_{I \times J} \leq \frac{1}{4}. \tag{3.4}$$

Let $T : H_N^1(\delta^2) \rightarrow H_N^1(\delta^2)$ be an operator such that $\|T\| \leq M$. Now apply Theorem 3.2 to T . This gives a block basis $\{b_{I \times J} : I \times J \in \mathcal{R}_{N_1}\}$ satisfying the

conclusions (i)–(iv) of Theorem 3.2. The Ramsey theorem 2.1 for colored dyadic rectangles applied to

$$\mathcal{C} = \{I \times J \in \mathcal{R}_{N_1} : |\langle Tb_{I \times J}, b_{I \times J} \rangle| \geq \|b_{I \times J}\|_2^2/2\}$$

yields collections $\mathcal{A}, \mathcal{B} \subset \mathcal{D}^{N_1}$, with Carleson constants $\llbracket \mathcal{A} \rrbracket \geq N_2$ and $\llbracket \mathcal{B} \rrbracket \geq N_2$, such that $\mathcal{A} \times \mathcal{B} \subset \mathcal{C}$ or $\mathcal{A} \times \mathcal{B} \subset \mathcal{R}_{N_1} \setminus \mathcal{C}$. We choose $H = T$ if $\mathcal{A} \times \mathcal{B} \subset \mathcal{C}$ and $H = \text{Id} - T$ if $\mathcal{A} \times \mathcal{B} \subset \mathcal{R}_{N_1} \setminus \mathcal{C}$.

The following lower estimate will be essential below:

$$|\langle Hb_{I \times J}, b_{I \times J} \rangle| \geq \|b_{I \times J}\|_2^2/2, \quad I \times J \in \mathcal{A} \times \mathcal{B}. \quad (3.5)$$

We define the product space $X_{\mathcal{A} \times \mathcal{B}}$ by

$$X_{\mathcal{A} \times \mathcal{B}} = \text{span}\{h_{I \times J} : I \times J \in \mathcal{A} \times \mathcal{B}\} \cap H^1(\delta^2).$$

Since $\llbracket \mathcal{A} \rrbracket \geq N_2$, $\llbracket \mathcal{B} \rrbracket \geq N_2$, we know from (2.2) that there exists a universal constant $C > 0$ so that

$$\begin{array}{ccc} H_n^1(\delta^2) & \xrightarrow{\text{Id}} & H_n^1(\delta^2) \\ E_0 \downarrow & & \uparrow P_0 \\ X_{\mathcal{A} \times \mathcal{B}} & \xrightarrow{\text{Id}} & X_{\mathcal{A} \times \mathcal{B}} \end{array} \quad \|E_0\| \|P_0\| \leq C.$$

We claim that Theorem 3.2 and the choices we made in (3.3), (3.4) and (3.5) imply that there exist linear operators S_1 and P_1 such that

$$\begin{array}{ccc} X_{\mathcal{A} \times \mathcal{B}} & \xrightarrow{\text{Id}} & X_{\mathcal{A} \times \mathcal{B}} \\ S_1 \downarrow & & \uparrow P_1 \\ H_N^1(\delta^2) & \xrightarrow{H} & H_N^1(\delta^2) \end{array} \quad \|E_1\| \|P_1\| \leq C,$$

for some universal constant $C > 0$. For the verification of the claim we remark that the method lined out in [17, 288–290] is directly applicable: The isomorphic embedding

$$S_1 : X_{\mathcal{A} \times \mathcal{B}} \rightarrow \text{span}\{b_{I \times J} : I \times J \in \mathcal{A} \times \mathcal{B}\}$$

is defined as the linear extension of the map

$$h_{I \times J} \mapsto b_{I \times J}.$$

For the norm estimate of S_1 we refer to (3.1). Next, define

$$\tilde{P}_1 : H_N^1(\delta^2) \rightarrow \text{span}\{b_{I \times J} : I \times J \in \mathcal{A} \times \mathcal{B}\}$$

by the formula

$$f \mapsto \sum_{I \times J \in \mathcal{A} \times \mathcal{B}} \langle f, b_{I \times J} \rangle b_{I \times J} \langle Hb_{I \times J}, b_{I \times J} \rangle^{-1},$$

and observe that $\|\tilde{P}_1\| \leq 2\|Q\|$. We observe that for $g \in \text{span}\{b_{I \times J} : I \times J \in \mathcal{A} \times \mathcal{B}\}$ we have

$$\tilde{P}_1 H g = g + G g,$$

where the error term Gg is controlled via $2 \sum_{I \times J \in \mathcal{R}_{N_1}} \varepsilon_{I \times J} \leq 1/2$ by the following operator norm estimate

$$\|G : \text{span}\{b_{I \times J} : I \times J \in \mathcal{A} \times \mathcal{B}\} \rightarrow \text{span}\{b_{I \times J} : I \times J \in \mathcal{A} \times \mathcal{B}\}\|_{H^1(\delta^2)} \leq \frac{1}{2}.$$

Hence, we may invert $\text{Id} + G$ on $\text{span}\{b_{I \times J} : I \times J \in \mathcal{A} \times \mathcal{B}\}$ so that

$$(\text{Id} + G)^{-1} \tilde{P}_1 H g = g, \quad g \in \text{span}\{b_{I \times J} : I \times J \in \mathcal{A} \times \mathcal{B}\}.$$

Note that $\|(\text{Id} + G)^{-1}\| \leq 2$. This defines P_1 as follows:

$$P_1 f = S_1^{-1}(\text{Id} + G)^{-1} \tilde{P}_1 f, \quad f \in X_{\mathcal{A} \times \mathcal{B}}.$$

We should emphasize that S_1^{-1} is well defined on the range of $(\text{Id} + G)^{-1}$ and furthermore $(\text{Id} + G)^{-1}$ is well defined on the range of \tilde{P}_1 .

Finally, it remains to merge the diagrams yielding the following factorization:

$$\begin{array}{ccc} H_n^1(\delta^2) & \xrightarrow{\text{Id}} & H_n^1(\delta^2) \\ E \downarrow & & \uparrow P \\ H_N^1(\delta^2) & \xrightarrow{H} & H_N^1(\delta^2) \end{array} \quad \|E\| \|P\| \leq C,$$

where $C > 0$ is a universal constant.

4. QUANTITATIVE ALMOST-DIAGONALIZATION

In this section we give the proof of Theorem 3.2. Our argument is inductive. We use induction within the collection of dyadic rectangles. It is therefore important that we introduce a suitable linear ordering relation on the collection of dyadic rectangles. Below we specifically construct the linear ordering relation \triangleleft so that the bijective index function $\mathcal{O}_{\triangleleft} : \mathcal{R} \rightarrow \mathbb{N}$, which is defined by

$$\mathcal{O}_{\triangleleft}(R_0) < \mathcal{O}_{\triangleleft}(R_1) \Leftrightarrow R_0 \triangleleft R_1, \quad R_0, R_1 \in \mathcal{R},$$

has the following properties (4.1) and (4.2). For a picture of the index function $\mathcal{O}_{\triangleleft}$ see Figure 1. The geometry of a dyadic rectangle and its position within our linear ordering \triangleleft are linked by the inequalities

$$(2^k - 1)^2 < \mathcal{O}_{\triangleleft}(I \times J) \leq (2^{k+1} - 1)^2, \quad \text{whenever } \min(|I|, |J|) = 2^{-k}, \quad (4.1)$$

as well as

$$4|I_1 \times J_1| \leq \frac{|I_0 \times J_0|}{\min(|I_1|, |J_1|)}, \quad \text{whenever } I_0 \times J_0 \triangleleft I_1 \times J_1. \quad (4.2)$$

Any linear orderings on the dyadic rectangles for which (4.1) and (4.2) hold may serve as basis for our induction argument in the proof of Theorem 3.2.

4.1. Constructing the linear ordering relation \triangleleft on \mathcal{R} .

First, we define the rectangles of fixed side lengths 2^{-m} and 2^{-n} by setting

$$\mathcal{B}_{m,n} = \{I \times J \in \mathcal{R} : |I| = 2^{-m}, |J| = 2^{-n}\}, \quad m, n \geq 0. \quad (4.3)$$

Second, we will define the ordering relation \prec_{ℓ} on each of the blocks $\mathcal{B}_{m,n}$. Given two dyadic rectangles $I_0 \times J_0, I_1 \times J_1 \in \mathcal{B}_{m,n}$ we set

$$I_0 \times J_0 \prec_{\ell} I_1 \times J_1 :\Leftrightarrow (\inf I_0, \inf J_0) <_{\ell} (\inf I_1, \inf J_1),$$

where $<_{\ell}$ denotes the lexicographic ordering on \mathbb{R}^2 . Third, we shall collect the blocks $\mathcal{B}_{m,n}$ in the collections

$$\mathcal{S}_k = \{\mathcal{B}_{m,n} : \max(m, n) = k\}, \quad k \geq 0.$$

Third, we need to bring the blocks $\mathcal{B}_{m,n}$ in order. To this end, we consider

$$w : \{\mathcal{B}_{m,n} : m, n \geq 0\} \rightarrow \mathbb{N}_0$$

such that the following conditions hold for all $k \geq 1$:

- (i) $w|_{\mathcal{S}_k} : \mathcal{S}_k \rightarrow \{k^2, \dots, (k+1)^2 - 1\}$ is bijective.

0		2		12	
				11	
		1		10	
				9	
3	4	6		28	32
				27	31
		5		26	30
				25	29
13	14	15	16	18	20
				22	24
				36	40
				44	48
17	19	21	23	35	39
				43	47
				34	38
				42	46
33	37	41	45	33	37
				41	45
				33	37
				41	45

FIGURE 1. Index function $\mathcal{O}_{\triangleleft}(I \times J)$ for all $I \times J \in \mathcal{R}_2$.

(ii) we set $w(\mathcal{B}_{0,k}) = k^2$ and moreover

$$w(\mathcal{B}_{m_0, n_0}) < w(\mathcal{B}_{m_1, n_1}) \Leftrightarrow \begin{cases} m_0 > n_0 \text{ and } m_1 \leq n_1, \\ m_0 > n_0 \text{ and } m_1 > n_1 \text{ and } n_0 < n_1, \\ m_0 \leq n_0 \text{ and } m_1 \leq n_1 \text{ and } m_0 < m_1, \end{cases}$$

for all $\mathcal{B}_{m_0, n_0}, \mathcal{B}_{m_1, n_1} \in \mathcal{S}_k \setminus \{\mathcal{B}_{0,k}\}$.

Finally, we use the function w and its properties as well as the properties of \prec_ℓ to define our linear ordering relation \triangleleft on the dyadic rectangles \mathcal{R} . If $I_0 \times J_0, I_1 \times J_1 \in \mathcal{R}$ we set

$$(I_0 \times J_0) \triangleleft (I_1 \times J_1) :\Leftrightarrow \begin{cases} w(\mathcal{B}_{\text{lev } I_0, \text{lev } J_0}) < w(\mathcal{B}_{\text{lev } I_1, \text{lev } J_1}) \text{ or} \\ w(\mathcal{B}_{\text{lev } I_0, \text{lev } J_0}) = w(\mathcal{B}_{\text{lev } I_1, \text{lev } J_1}) \text{ and } (I_0, J_0) \prec_\ell (I_1, J_1). \end{cases}$$

Since our ordering relation \triangleleft is linear, we may well define the bijective index function $\mathcal{O}_{\triangleleft} : \mathcal{R} \rightarrow \mathbb{N}$ by the following property:

$$\mathcal{O}_{\triangleleft}(R_0) < \mathcal{O}_{\triangleleft}(R_1) \Leftrightarrow R_0 \triangleleft R_1, \quad R_0, R_1 \in \mathcal{R}.$$

Observe that the crucial relations between the geometry of a dyadic rectangle and its position within our linear ordering (4.1) and (4.2) are satisfied by design.

4.2. Combinatorial lemma.

Let $\{r_i\}$ denote the sequence of independent Rademacher functions which are given by

$$r_i(t) = \text{sign}(\sin(2\pi 2^i t)), \quad t \in [0, 1], i \in \mathbb{N}.$$

We consider the tensor product $r_{i,j}$ of the standard Rademacher system defined as

$$r_{i,j}(s, t) = r_i(s) r_j(t), \quad (s, t) \in [0, 1]^2$$

It is well known and easy to verify that in both spaces, $H^1(\delta^2)$ and $BMO(\delta^2)$, the system $\{r_{i,j}\}$ is equivalent to the unit vector basis of ℓ^2 . Specifically, there exists constants c_0, C_0 so that for any sequence of scalars $\{a_{i,j}\}$ the following inequalities hold.

$$\left\| \sum a_{i,j} r_{i,j} \right\|_{H^1(\delta^2)}^2 = \sum a_{i,j}^2$$

and

$$c_0 \sum a_{i,j}^2 \leq \left\| \sum a_{i,j} r_{i,j} \right\|_{BMO(\delta^2)}^2 \leq C_0 \sum a_{i,j}^2.$$

Hence, $\{r_{ij}\}$ is a weak null sequence in both spaces $H^1(\delta^2)$ and $BMO(\delta^2)$,

$$r_{ij} \rightarrow 0 \quad \text{weakly in } H^1(\delta^2), \text{ if } i \rightarrow \infty \text{ or } j \rightarrow \infty$$

and

$$r_{ij} \rightarrow 0 \quad \text{weakly in } BMO(\delta^2), \text{ if } i \rightarrow \infty \text{ or } j \rightarrow \infty.$$

For the purpose of our present work we need a quantitative strengthening of these considerations. This is done in the following combinatorial lemma. Our combinatorial argument is controlled by the local frequency weight

$$f(K \times L) = |\langle x, h_{K \times L} \rangle| + |\langle y, h_{K \times L} \rangle|, \quad K \times L \subset K_0 \times L_0$$

where $x \in BMO(\delta^2)$ and $y \in H^1(\delta^2)$ are fixed functions and $K_0 \times L_0 \in \mathcal{R}$. For us, it will be extremely important that the collection

$$\{K \times L : f(K \times L) \leq \tau |K \times L|\}$$

contains almost complete and well-structured coverings of $K_0 \times L_0$ of the form

$$\{K_0 \times L : L \in \mathcal{D}_\ell \cap L_0\} \quad \text{and} \quad \{K \times L_0 : K \in \mathcal{D}_k \cap K_0\},$$

with k and ℓ well under control in terms of τ . See Figure 3.

Lemma 4.1. *Let $i \in \mathbb{N}$, $K_0, L_0 \in \mathcal{D}$, $x_j \in BMO(\delta^2)$, $y_j \in H^1(\delta^2)$, $1 \leq j \leq i$, such that*

$$\sum_{j=1}^i \|x_j\|_{BMO(\delta^2)} \leq 1 \quad \text{and} \quad \sum_{j=1}^i \|y_j\|_{H^1(\delta^2)} \leq |K_0 \times L_0|. \quad (4.4)$$

Let $\tau > 0$, $r \in \mathbb{N}_0$, $K \times L \in \mathcal{R}$ and define the local frequency weight

$$f_i(K \times L) = \sum_{j=1}^i |\langle x_j, h_{K \times L} \rangle| + |\langle y_j, h_{K \times L} \rangle| \quad (4.5)$$

as well as the collections

$$\mathcal{K}(K_0 \times L_0) = \{K \times L_0 : K \subset K_0, |K| \leq 2^{-r}|K_0|, f_i(K \times L_0) \leq \tau |K \times L_0|\},$$

$$\mathcal{L}(K_0 \times L_0) = \{K_0 \times L : L \subset L_0, |L| \leq 2^{-r}|L_0|, f_i(K_0 \times L) \leq \tau |K_0 \times L|\}.$$

For all integers k, ℓ the collections $\mathcal{K}_k(K_0 \times L_0)$ and $\mathcal{L}_\ell(K_0 \times L_0)$ are given by

$$\mathcal{K}_k(K_0 \times L_0) = \mathcal{K}(K_0 \times L_0) \cap (\{K \in \mathcal{D} : |K| = 2^{-k}|K_0|\} \times \mathcal{D}),$$

$$\mathcal{L}_\ell(K_0 \times L_0) = \mathcal{L}(K_0 \times L_0) \cap (\mathcal{D} \times \{L \in \mathcal{D} : |L| = 2^{-\ell}|L_0|\}).$$

Let $\delta > 0$. Then there exist integers k, ℓ with

$$r \leq k, \ell \leq \lfloor \frac{i^2}{\delta^2 \tau^2} \rfloor + r \quad (4.6)$$

such that

$$|\mathcal{K}_k^*(K_0 \times L_0)| \geq (1 - \delta)|K_0 \times L_0| \quad \text{and} \quad |\mathcal{L}_\ell^*(K_0 \times L_0)| \geq (1 - \delta)|K_0 \times L_0|. \quad (4.7)$$

Proof. Define $\mathcal{B} = \{K \times L_0 : K \subset K_0\} \setminus \mathcal{K}(K_0 \times L_0)$ and

$$\mathcal{B}_k = \mathcal{B} \cap (\{K \in \mathcal{D} : |K| = 2^{-k}|K_0|\} \times \mathcal{D}),$$

see Figure 2. Let

$$A = \lfloor \frac{i^2}{\delta^2 \tau^2} \rfloor + r.$$

By construction \mathcal{B}_k and $\mathcal{K}_k(K_0 \times L_0)$ form a disjoint decomposition of $K_0 \times L_0$. We will determine a collection $\mathcal{K}_k(K_0 \times L_0)$ by showing that \mathcal{B}_k^* is small enough for at least one value of k . Now assume the opposite, namely that

$$|\mathcal{B}_k^*| \geq \delta |K_0 \times L_0|, \quad r \leq k \leq A.$$

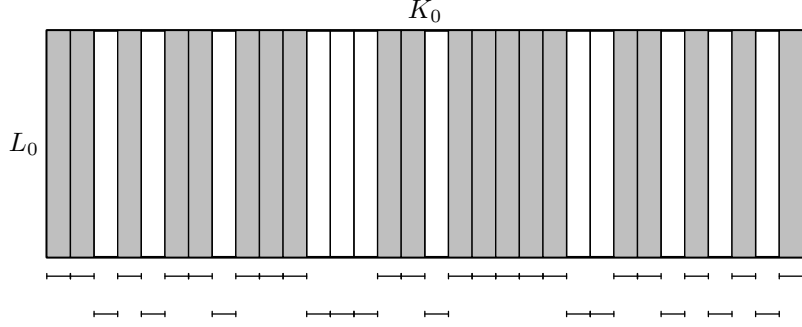


FIGURE 2. The shaded rectangles form $\mathcal{K}_k(K_0 \times L_0)$, the white rectangles form \mathcal{B}_k . The x -component of $\mathcal{K}_k(K_0 \times L_0)$ (first level of intervals) and the x -component of \mathcal{B}_k (second level of intervals) form a disjoint cover of K_0 .

Summing these estimates yields

$$\sum_{k=r}^A |\mathcal{B}_k^*| \geq (A - r + 1) \delta |K_0 \times L_0|, \quad (4.8)$$

Observe that

$$\begin{aligned} \tau \cdot \sum_{k=r}^A |\mathcal{B}_k^*| &\leq \sum_{j=1}^i \sum_{k=r}^A \sum_{K \times L_0 \in \mathcal{B}_k} |\langle x_j, h_{K \times L_0} \rangle| + |\langle y_j, h_{K \times L_0} \rangle| \\ &= \sum_{j=1}^i |\langle x_j, \sum_{k=r}^A \sum_{K \times L_0 \in \mathcal{B}_k} \pm h_{K \times L_0} \rangle| + |\langle y_j, \sum_{k=r}^A \sum_{K \times L_0 \in \mathcal{B}_k} \pm h_{K \times L_0} \rangle|. \end{aligned}$$

By (2.5) we have

$$\begin{aligned} \left\| \sum_{k=r}^A \sum_{K \times L_0 \in \mathcal{B}_k} \pm h_{K \times L_0} \right\|_{H^1(\delta^2)} &= \sqrt{A - r + 1} |K_0 \times L_0|, \\ \left\| \sum_{k=r}^A \sum_{K \times L_0 \in \mathcal{B}_k} \pm h_{K \times L_0} \right\|_{BMO(\delta^2)} &= \sqrt{A - r + 1}, \end{aligned}$$

thus, by duality and (4.4) we obtain

$$\tau \cdot \sum_{k=r}^A |\mathcal{B}_k^*| \leq i \sqrt{A - r + 1} |K_0 \times L_0|. \quad (4.9)$$

Combining (4.8) and (4.9) we conclude

$$A \leq \frac{i^2}{\delta^2 \tau^2} + r - 1,$$

which contradicts the definition of A . Thus we found $r \leq k \leq A$ so that

$$|\mathcal{K}_k^*(K_0 \times L_0)| \geq (1 - \delta) |K_0 \times L_0|,$$

see Figure 2. We emphasize that the x -component of the rectangles in $\mathcal{K}_k^*(K_0 \times L_0)$ are covering a set of measure $\geq (1 - \delta) |K_0|$ in K_0 . The y -component of the rectangles in $\mathcal{K}_k^*(K_0 \times L_0)$ equals L_0 throughout. (For the later use of this lemma it is extremely important that we found a large collection of rectangles $\mathcal{K}_k(K_0 \times L_0)$ where the y -component L_0 remains intact.) See Figure 3.

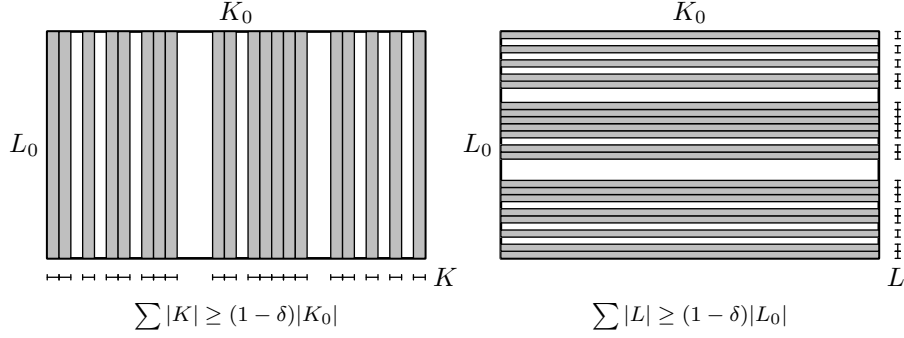


FIGURE 3. Two good covers. The shaded rectangles in the left picture are from $\mathcal{K}_k(K_0 \times L_0)$. For those rectangles the y -component L_0 remains intact and the x -components K form a large cover of K_0 . **Therefore** the construction in Section 4.3 yields the crucial measure estimate (4.48). The right picture displays the collection $\mathcal{L}_\ell(K_0 \times L_0)$ and the roles of x and y -components are interchanged.

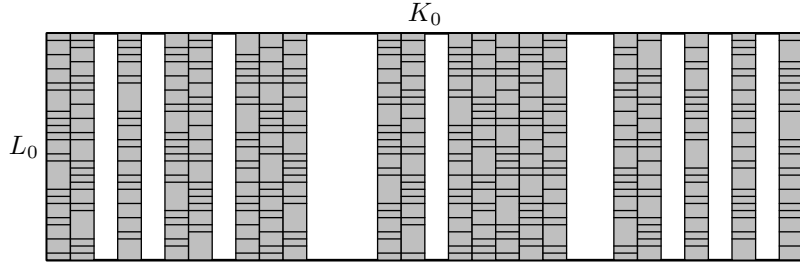


FIGURE 4. A bad cover of $K_0 \times L_0$. These fragmented shaded rectangles cover the same subset of $K_0 \times L_0$ as $\mathcal{K}_k(K_0 \times L_0)$ (see Figure 3). The y -component L_0 did not remain intact and **therefore** the construction in Section 4.3 would not yields the crucial measure estimate (4.48).

The same proof in the other variable can be used to show the estimate for \mathcal{L}_ℓ^* . \square

4.3. Proof of Theorem 3.2.

Theorem 3.2 asserts that we are able to construct a large block basis $\{b_{I \times J}\}$ in $H_N^1(\delta^2)$ which are almost eigenvectors for T . Moreover, the block basis is such that it spans a well complemented copy of $H_n^1(\delta^2)$ in $H_N^1(\delta^2)$. We choose the normalization $M = 1$ and $\|T\| \leq 1$.

It is here where we will exploit our linear order \triangleleft introduced on the collection of dyadic rectangles \mathcal{R} . The proof described below is by mathematical induction executed along the linear order given by $\mathcal{O}_{\triangleleft}$.

Inductive construction.

To make the transition from standard indexing by dyadic rectangles to indexing by natural numbers we employ the following convention. Given a dyadic rectangle

$I \times J$ with $\mathcal{O}_\triangleleft(I \times J) = i$ we will systematically relabel the collections $\mathcal{E}_{I \times J}$, the functions $b_{I \times J}$ and the constants $\delta_{I \times J}$, $\tau_{I \times J}$ by \mathcal{E}_i , b_i and δ_i , τ_i , respectively.

Before we begin with our construction we explicitly define the constants δ_i by

$$\delta_i = 2^{-i}/(8n). \quad (4.10)$$

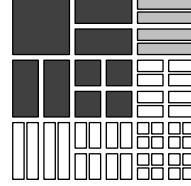
The remaining crucial constants τ_i will be defined inductively as the construction proceeds.

First stage of the induction. We begin the induction by setting $\mathcal{E}_1 := \mathcal{E}_{[0,1] \times [0,1]} := \{[0,1] \times [0,1]\}$ and $b_1 := b_{[0,1] \times [0,1]} := h_{[0,1] \times [0,1]}$.

At stage i of the induction. We assume that we have already defined the disjoint collections of dyadic rectangles \mathcal{E}_j for all $1 \leq j \leq i-1$. Now, we will construct \mathcal{E}_i . The construction of \mathcal{E}_i depends crucially on the value of i . We will distinguish between two principal cases, where the second one is divided again into two sub cases.

- ▷ Case 1: The stage ordinal i is given by $i = \mathcal{O}_\triangleleft([0,1] \times J)$.
- ▷ Case 2: The stage ordinal i is given by $i = \mathcal{O}_\triangleleft(I \times J)$, where $I \neq [0,1]$.
 - + Case 2.a: The second component J satisfies $J = [0,1]$.
 - + Case 2.b: The second component J satisfies $J \neq [0,1]$.

Case 1: $I = [0,1]$. The stage ordinal i is given by $i = \mathcal{O}_\triangleleft([0,1] \times J)$. Case 1 is applicable to the light rectangles. The collections $\mathcal{E}_{I_0 \times J_0}$ indexed by the dark rectangles $I_0 \times J_0$ are already well defined at this stage. The white ones are ignored.



Recall that

$$b_j = \sum_{K \times L \in \mathcal{E}_j} h_{K \times L}, \quad 1 \leq j \leq i-1.$$

Since the collection \mathcal{E}_j consists of pairwise disjoint rectangles we have by (1.1) and duality that

$$\|b_j\|_{BMO(\delta^2)} = 1 \quad \text{and} \quad \|b_j\|_{H^1(\delta^2)} = |\mathcal{E}_j^*|.$$

Let \tilde{J} denote the unique dyadic interval satisfying $\tilde{J} \supset J$ and $|\tilde{J}| = 2|J|$. By definition of our linear ordering we have $\mathcal{O}_\triangleleft([0,1] \times \tilde{J}) \leq i-1$. Hence, $\mathcal{E}_{[0,1] \times \tilde{J}}$ is already defined. Now put

$$\beta_i = \min\{|K_0 \times L_0| : K_0 \times L_0 \in \mathcal{E}_{[0,1] \times \tilde{J}}\} \quad (4.11)$$

and define for all $1 \leq j \leq i-1$

$$x_j := \frac{1}{i-1} T^* b_j, \quad y_j := \frac{\beta_i}{(i-1)|\mathcal{E}_j^*|} T b_j, \quad (4.12)$$

Recall that we are using the normalization $\|T\| \leq 1$, hence

$$\sum_{j=1}^{i-1} \|x_j\|_{BMO(\delta^2)} \leq 1 \quad \text{and} \quad \sum_{j=1}^{i-1} \|y_j\|_{H^1(\delta^2)} \leq \beta_i.$$

We define the local frequency weight

$$f_{i-1}(K \times L) = \sum_{j=1}^{i-1} |\langle x_j, h_{K \times L} \rangle| + |\langle y_j, h_{K \times L} \rangle|, \quad K \times L \in \mathcal{R}. \quad (4.13)$$

Given L_0 we remark that by our previous choices we have the following convenient implication:

$$K \times L_0 \in \mathcal{E}_{[0,1] \times \tilde{J}} \quad \text{implies} \quad K = [0,1]. \quad (4.14)$$

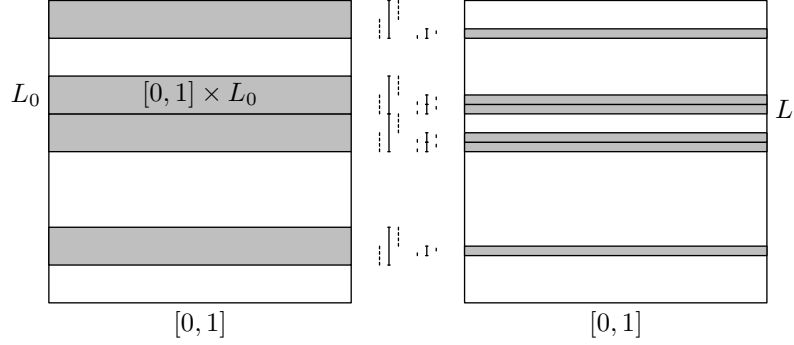


FIGURE 5. This figure displays the transition from $\mathcal{E}_{[0,1] \times \tilde{J}}$ to $\mathcal{E}_{[0,1] \times J}$ by means of a generic Gamlen-Gaudet step (4.19). The shaded rectangles $[0, 1] \times L_0$ on the left form $\mathcal{E}_{[0,1] \times \tilde{J}}$, the shaded rectangles $[0, 1] \times L$ on the right form $\mathcal{E}_{[0,1] \times J}$. The union of the rectangles in $\mathcal{E}_{[0,1] \times J}$ is contained in $E_{[0,1] \times \tilde{J}}^\ell$, since in this figure J is the left half of \tilde{J} . The center displays the y -components of $b_{[0,1] \times \tilde{J}}$ (center left) and of $b_{[0,1] \times J}$ (center right).

We now define the constant τ_i by

$$\tau_i = \frac{2^{-i}}{4(i-1)} \beta_i \min_{j \leq i} \varepsilon_j \|b_j\|_2^2 \quad (4.15)$$

For all L_0 such that $[0, 1] \times L_0 \in \mathcal{E}_{[0,1] \times \tilde{J}}$, we define the collection of dyadic rectangles

$$\mathcal{L}([0, 1] \times L_0) = \{[0, 1] \times L : L \subsetneq L_0, f_{i-1}([0, 1] \times L) \leq \tau_i |L|\}.$$

Applying Lemma 4.1 to $\mathcal{L}([0, 1] \times L_0)$ yields an integer $\ell = \ell([0, 1] \times L_0)$ so that

$$1 \leq \ell([0, 1] \times L_0) < \frac{(i-1)^2}{\delta_i^2 \tau_i^2} + 1 \quad (4.16)$$

such that the collection of disjoint dyadic rectangles

$$\mathcal{Z}_{[0,1] \times J}([0, 1] \times L_0) = \{[0, 1] \times L \in \mathcal{L}([0, 1] \times L_0) : |L| = 2^{-\ell([0,1] \times L_0)} |L_0|\}$$

satisfies the estimate

$$(1 - \delta_i) |[0, 1] \times L_0| \leq |\mathcal{Z}_{[0,1] \times J}^*([0, 1] \times L_0)| \leq |[0, 1] \times L_0|. \quad (4.17)$$

Note that in Lemma 4.1 $\mathcal{Z}_{[0,1] \times J}([0, 1] \times L_0)$ was denoted $\mathcal{Z}_\ell([0, 1] \times L_0)$. Now we take the union and define

$$\mathcal{Z}_{[0,1] \times J} = \bigcup \{ \mathcal{Z}_{[0,1] \times J}([0, 1] \times L_0) : [0, 1] \times L_0 \in \mathcal{E}_{[0,1] \times \tilde{J}} \}.$$

Since $\mathcal{Z}_{[0,1] \times J}([0, 1] \times L_0) \subset \mathcal{L}([0, 1] \times L_0)$, we know

$$f_{i-1}([0, 1] \times L) \leq \tau_i |L|, \quad \text{for } [0, 1] \times L \in \mathcal{Z}_{[0,1] \times J}. \quad (4.18)$$

We are now ready to define $\mathcal{E}_{[0,1] \times J}$ using the Gamlen-Gaudet procedure. To this end recall first that \tilde{J} denotes the unique dyadic interval satisfying $\tilde{J} \supset J$ and $|\tilde{J}| = 2|J|$. For a dyadic interval L_0 we denote its left half by L_0^ℓ ($L_0^\ell \subset L_0$, $|L_0^\ell| = |L_0|/2$, $\inf L_0^\ell = \inf L_0$) and its right half by L_0^r ($L_0^r = L_0 \setminus L_0^\ell$). We define the sets

$$E_{[0,1] \times \tilde{J}}^\ell = \bigcup_{[0,1] \times L_0 \in \mathcal{E}_{[0,1] \times \tilde{J}}} [0, 1] \times L_0^\ell \quad \text{and} \quad E_{[0,1] \times \tilde{J}}^r = \bigcup_{[0,1] \times L_0 \in \mathcal{E}_{[0,1] \times \tilde{J}}} [0, 1] \times L_0^r.$$

If J is the left half of \tilde{J} , then we put

$$\mathcal{E}_{[0,1] \times J} = \{[0,1] \times L \in \mathcal{Z}_{[0,1] \times J} : [0,1] \times L \subset E_{[0,1] \times \tilde{J}}^\ell\}. \quad (4.19a)$$

See Figure 5. In the other case when J is the right half of \tilde{J} we put accordingly

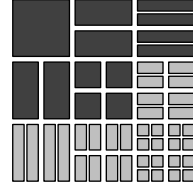
$$\mathcal{E}_{[0,1] \times J} = \{[0,1] \times L \in \mathcal{Z}_{[0,1] \times J} : [0,1] \times L \subset E_{[0,1] \times \tilde{J}}^r\}. \quad (4.19b)$$

Recall that $i = \mathcal{O}_\triangleleft([0,1] \times J)$ and $\delta_i = \delta_{[0,1] \times J}$. An immediate consequence of the Gamlen-Gaudet construction and (4.17) is the estimate

$$\frac{1}{2}(1 - \delta_{[0,1] \times J})|[0,1] \times L| \leq |[0,1] \times L| \cap \mathcal{E}_{[0,1] \times J}^* \leq \frac{1}{2}|[0,1] \times L|, \quad (4.20)$$

for all $[0,1] \times L \in \mathcal{E}_{[0,1] \times \tilde{J}}$. Note that all the rectangles in $\mathcal{E}_{[0,1] \times \tilde{J}}$ are of the form $[0,1] \times L$, see (4.14).

Case 2: $I \neq [0,1]$. The figure on the right depicts the transition from Case 1 to Case 2. Here, the stage ordinal i is given by $i = \mathcal{O}_\triangleleft(I \times J)$ with $I \neq [0,1]$. The rectangle $I \times J$ is one of the light rectangles. The light rectangles fall into two separate cases, see below. Up to (4.33) both cases are treated in tandem.



We will now construct the collections $\mathcal{Y}_{I \times J}$ of y -frequencies and depending on each y -frequency $L_0 \in \mathcal{Y}_{I \times J}$ the collection $\mathcal{X}_{I \times J}(L_0)$ of x -frequencies. These frequencies will be our building blocks for $\mathcal{E}_{I \times J}$ and we have

$$\mathcal{E}_{I \times J} \subset \bigcup \{ \mathcal{X}_{I \times J}(L_0) \times \{L_0\} : L_0 \in \mathcal{Y}_{I \times J} \}.$$

The rules by which we finally select $\mathcal{E}_{I \times J}$ from the above large union are given in the equations (4.33).

First, let us define the collection $\mathcal{Y}_{I \times J}$ simply by putting

$$\mathcal{Y}_{I \times J} = \{L_0 : [0,1] \times L_0 \in \mathcal{E}_{[0,1] \times J}\}.$$

We remark that $K \times L_0 \in \mathcal{E}_{[0,1] \times J}$ implies $K = [0,1]$. Fix $L_0 \in \mathcal{Y}_{I \times J}$. By means of the combinatorial Lemma 4.1 we will construct the collection $\mathcal{X}_{I \times J}(L_0)$ of dyadic intervals so that $\mathcal{X}_{I \times J}(L_0) \times \{L_0\}$ is an almost complete cover of $[0,1] \times L_0$ and simultaneously the rectangles in $\mathcal{X}_{I \times J}(L_0) \times \{L_0\}$ have almost vanishing local frequency weight (4.13).

Now, let \mathcal{P} denote the previous dyadic rectangle indices that are not located in the same macro block $\mathcal{B}_{m,n}$ as $I \times J$, see (4.3). That is

$$\mathcal{P} = \{I_0 \times J_0 : I_0 \times J_0 \triangleleft I \times J, (|I_0|, |J_0|) \neq (|I|, |J|), |I_0| \geq |I|, |J_0| \geq |J|\},$$

see Figure 6. For each $I_0 \times J_0 \in \mathcal{P}$ there exists a unique $L'_0 \in \mathcal{Y}_{I_0 \times J_0}$ such that $L'_0 \cap L_0 \neq \emptyset$ in which case $L'_0 \supset L_0$, see Figure 6. We display the logical dependence by writing

$$L'_0 = L'_0(I \times J, L_0, I_0 \times J_0). \quad (4.21)$$

Next, we further partition \mathcal{P} into strip collections. Recall that $\mathcal{D}_m = \{I \in \mathcal{D} : |I| = 2^{-m}\}$. We define \mathbb{A} by the following rule: if $I_0 \times J_0 \in \mathcal{P}$ and $|I_0| = 2^{-m}$ we put $\mathcal{D}_m \times \{J_0\} \in \mathbb{A}$. In other words

$$\mathbb{A} = \{\mathcal{D}_m \times \{J_0\} : m \in \mathbb{N}_0, I_0 \times J_0 \in \mathcal{P}, I_0 \in \mathcal{D}_m\}. \quad (4.22)$$

See Figure 6. Note that if $\mathcal{S} \in \mathbb{A}$ then there exist m and $J_0 \in \mathcal{D}$ such that $\mathcal{S} = \mathcal{D}_m \times \{J_0\}$ and for each $I'_0 \in \mathcal{D}_m$ we have $I'_0 \times J_0 \in \mathcal{P}$. We have clearly $\mathcal{P} = \bigcup_{\mathcal{S} \in \mathbb{A}} \mathcal{S}$.

After the preparatory step above we now turn to the core construction, which uses $\mathcal{X}_{I_0 \times J_0}(L'_0)$, $I_0 \times J_0 \in \mathcal{P}$ as input and returns the collection $\mathcal{X}_{I \times J}(L_0)$ as

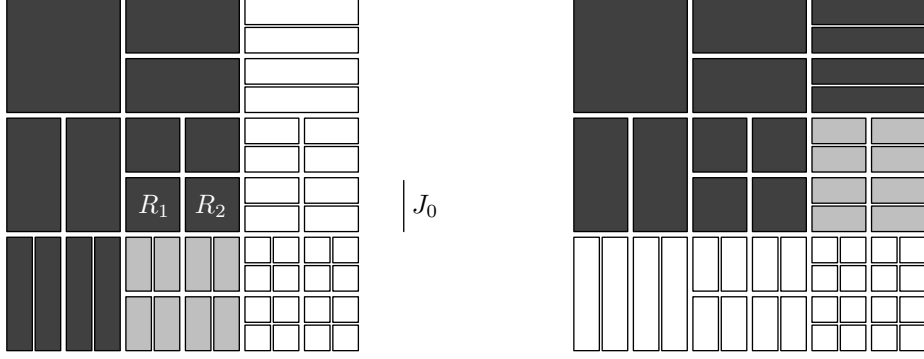


FIGURE 6. If $I \times J$ is one of the shaded rectangles then \mathcal{P} is the collection of the black rectangles for two generic situations. If L'_0 is defined by (4.21) then $L'_0 \cap L_0 \neq \emptyset$ and $L'_0 \subsetneq L_0$ is excluded by choice of \mathcal{P} . The interval J_0 defines the strip $\mathcal{D}_1 \times \{J_0\} = \{R_1, R_2\}$ in \mathbb{A} , see (4.22).

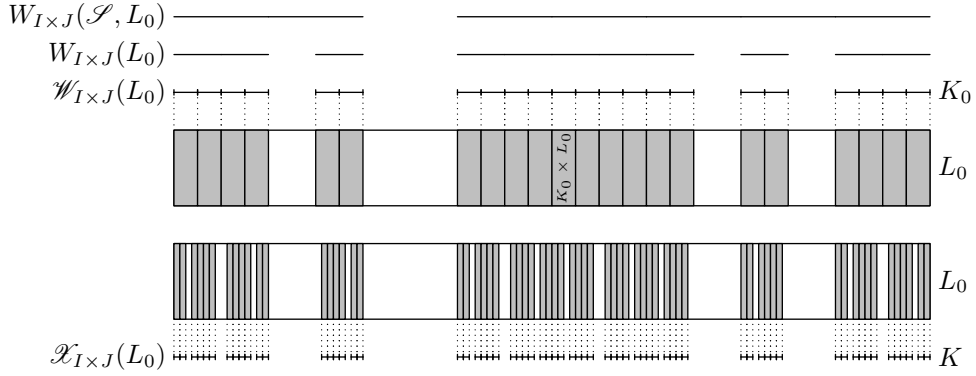


FIGURE 7. From top to bottom the figure describes the transition from the input data $W_{I \times J}(\mathcal{S}, L_0)$ to their intersection $W_{I \times J}(L_0)$ to the fine covering $\mathcal{W}_{I \times J}(L_0)$ and the rectangles $\mathcal{W}_{I \times J}(L_0) \times \{L_0\}$. The refinement $\mathcal{X}_{I \times J}(L_0) \times \{L_0\}$ of $\mathcal{W}_{I \times J}(L_0) \times \{L_0\}$ results from Lemma 4.1, hence $(K_0 \cap \mathcal{X}_{I \times J}(L_0)) \times \{L_0\}$ almost covers $K_0 \times L_0$.

output. We extract the relevant information carried by the input collection by defining the following sets:

$$W_{I \times J}(\mathcal{S}, L_0) = \bigcup \{ \mathcal{X}_{I_0 \times J_0}^*(L'_0) : I_0 \times J_0 \in \mathcal{S} \}, \quad \mathcal{S} \in \mathbb{A}. \quad (4.23)$$

(We emphasize the logical dependence $L'_0 = L'_0(I \times J, L_0, I_0 \times J_0)$.) See Figure 7. Now we take the intersection over the strips $\mathcal{S} \in \mathbb{A}$

$$W_{I \times J}(L_0) = \bigcap_{\mathcal{S} \in \mathbb{A}} W_{I \times J}(\mathcal{S}, L_0).$$

We next choose a fine covering of $W_{I \times J}(L_0)$ by intervals of equal length. To this we put

$$\eta_i = \frac{1}{2} \min \{ |K| : \exists L, K \times L \in \bigcup_{I_0 \times J_0 \in \mathcal{P}} \mathcal{E}_{I_0 \times J_0} \},$$

and set

$$\mathcal{W}_{I \times J}(L_0) = \{ K \in \mathcal{D} : |K| = \eta_i, K \subset W_{I \times J}(L_0) \}.$$

The collection of intervals $\mathcal{W}_{I \times J}(L_0)$ gives rise to the collection of pairwise disjoint dyadic rectangles $\mathcal{W}_{I \times J}(L_0) \times \{L_0\}$, see Figure 7. By means of the combinatorial Lemma 4.1 we refine this collection of rectangles and obtain a almost complete covering of $W_{I \times J}(L_0) \times \{L_0\}$ consisting of rectangles $K \times L_0$ with almost vanishing local frequency weight f_{i-1} specified below. It is important that in the refined covering the y -component L_0 remains intact.

We defined previously that

$$b_j = \sum_{K \times L \in \mathcal{E}_j} h_{K \times L}, \quad 1 \leq j \leq i-1.$$

Since the collection \mathcal{E}_j consists of pairwise disjoint rectangles we have by (1.1) and duality that

$$\|b_j\|_{BMO(\delta^2)} = 1 \quad \text{and} \quad \|b_j\|_{H^1(\delta^2)} = |\mathcal{E}_j^*|.$$

Now, put

$$\beta_i = \min\{|K_0 \times L_0| : L_0 \in \mathcal{Y}_{I \times J}, K_0 \in \mathcal{W}_{I \times J}(L_0)\} \quad (4.24)$$

and define for all $1 \leq j \leq i-1$

$$x_j := \frac{1}{i-1} T^* b_j, \quad y_j := \frac{\beta_i}{(i-1)|\mathcal{E}_j^*|} T b_j, \quad (4.25)$$

Recall that we are using the normalization $\|T\| \leq 1$, hence

$$\sum_{j=1}^{i-1} \|x_j\|_{BMO(\delta^2)} \leq 1 \quad \text{and} \quad \sum_{j=1}^{i-1} \|y_j\|_{H^1(\delta^2)} \leq \beta_i.$$

We next define the local frequency weight

$$f_{i-1}(K \times L) = \sum_{j=1}^{i-1} |\langle x_j, h_{K \times L} \rangle| + |\langle y_j, h_{K \times L} \rangle|, \quad K \times L \in \mathcal{R}. \quad (4.26)$$

We fix $K_0 \in \mathcal{W}_{I \times J}(L_0)$ and let

$$\mathcal{K}(K_0 \times L_0) = \{K \times L_0 : K \subsetneq K_0, f_{i-1}(K \times L_0) \leq \tau_i |K \times L_0|\},$$

where the constant τ_i is given by

$$\tau_i = \frac{2^{-i}}{4(i-1)} \beta_i \min_{j \leq i} \varepsilon_j \|b_j\|_2^2. \quad (4.27)$$

Applying Lemma 4.1 to $\mathcal{K}(K_0 \times L_0)$ yields an integer $k = k(K_0 \times L_0)$ such that

$$1 \leq k(K_0 \times L_0) < \frac{(i-1)^2}{\delta_i^2 \tau_i^2} + 1 \quad (4.28)$$

and so that

$$\mathcal{Z}_{I \times J}(K_0 \times L_0) = \{K \times L_0 \in \mathcal{K}(K_0 \times L_0) : |K| = 2^{-k(K_0 \times L_0)} |K_0|\}$$

satisfies

$$(1 - \delta_i) |K_0 \times L_0| \leq |\mathcal{Z}_{I \times J}^*(K_0 \times L_0)| \leq |K_0 \times L_0|. \quad (4.29)$$

Finally, the result of our construction is thus

$$\mathcal{X}_{I \times J}(L_0) = \{K : K \times L_0 \in \mathcal{Z}_{I \times J}(K_0 \times L_0), K_0 \in \mathcal{W}_{I \times J}(L_0)\}, \quad (4.30)$$

and

$$\mathcal{Z}_{I \times J} = \bigcup \{\mathcal{X}_{I \times J}(L_0) \times \{L_0\} : L_0 \in \mathcal{Y}_{I \times J}\}. \quad (4.31)$$

Observe that the following identity holds:

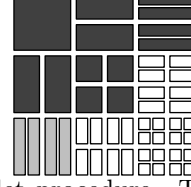
$$\mathcal{Z}_{I \times J} = \bigcup \{\mathcal{Z}_{I \times J}(K_0 \times L_0) : L_0 \in \mathcal{Y}_{I \times J}, K_0 \in \mathcal{W}_{I \times J}(L_0)\}.$$

Since $\mathcal{Z}_{I \times J}(K_0 \times L_0) \subset \mathcal{K}(K_0 \times L_0)$, we have the estimate

$$f_{i-1}(K \times L_0) \leq \tau_i |K \times L_0|, \quad \text{for all } K \times L_0 \in \mathcal{Z}_{I \times J}. \quad (4.32)$$

Up to this point, the construction for Case 2.a and Case 2.b are identical. Now is the time to distinguish between the cases $J = [0, 1]$ and $J \neq [0, 1]$.

Case 2.a: $I \neq [0, 1]$, $J = [0, 1]$. The light rectangles $I \times J$ are the ones to which Case 2.a is applicable. The collection $\mathcal{E}_{I \times J}$ is defined in (4.33a). The collections $\mathcal{E}_{I_0 \times J_0}$ indexed by the dark rectangles $I_0 \times J_0$ are already well defined. The white ones are ignored.



We are now ready to define $\mathcal{E}_{I \times J}$ using the Gamlen-Gaudet procedure. To this end recall first that \tilde{I} denotes the unique dyadic interval satisfying $\tilde{I} \supset I$ and $|\tilde{I}| = 2|I|$. Second, for a dyadic interval K_0 we denote its left half by K_0^ℓ ($K_0^\ell \subset K_0$, $|K_0^\ell| = |K_0|/2$, $\inf K_0^\ell = \inf K_0$) and its right half by K_0^r ($K_0^r = K_0 \setminus K_0^\ell$). We define the sets

$$E_{\tilde{I} \times J}^\ell = \bigcup_{K_0 \times L_0 \in \mathcal{E}_{\tilde{I} \times J}} K_0^\ell \times L_0 \quad \text{and} \quad E_{\tilde{I} \times J}^r = \bigcup_{K_0 \times L_0 \in \mathcal{E}_{\tilde{I} \times J}} K_0^r \times L_0.$$

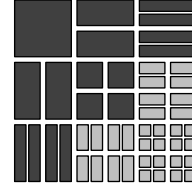
If I is the left half of \tilde{I} , then we put

$$\mathcal{E}_{I \times J} = \{K \times L_0 \in \mathcal{Z}_{I \times J} : K \times L_0 \subset E_{\tilde{I} \times J}^\ell\}, \quad (4.33a)$$

Alternatively, if I is the right half of \tilde{I} , then

$$\mathcal{E}_{I \times J} = \{K \times L_0 \in \mathcal{Z}_{I \times J} : K \times L_0 \subset E_{\tilde{I} \times J}^r\}. \quad (4.33b)$$

Case 2.b: $I \neq [0, 1]$, $J \neq [0, 1]$. The figure on the right depicts the transition from Case 2.a to Case 2.b. The light rectangles $I \times J$ are the ones covered by Case 2.b. The collection $\mathcal{E}_{I \times J}$ is defined in (4.33c). The collections $\mathcal{E}_{I_0 \times J_0}$ indexed by the dark rectangles $I_0 \times J_0$ are well defined before the first light rectangle is treated.



$$\mathcal{E}_{I \times J} = \{K \times L_0 \in \mathcal{Z}_{I \times J} : K \times L_0 \subset \mathcal{E}_{I \times \tilde{J}}^*\} \quad (4.33c)$$

(A comment on (4.33c): It is here where we bring in the combinatorial harvest of Lemma 4.1 where we insisted that the coverings leave the y -components L_0 intact, see Figure 3. Moreover, the definition (4.33c) would not be possible if we used fragmented coverings as depicted in Figure 4.)

In each of the above cases (4.33) we put

$$b_{I \times J} = \sum_{K \times L \in \mathcal{E}_{I \times J}} h_{K \times L}. \quad (4.34)$$

A first property of $\mathcal{E}_{I \times J}$.

We have now completed the construction part of the proof. Before we turn to a detailed examination of the entire system $\{\mathcal{E}_{I \times J} : I \times J \in \mathcal{R}\}$ and $\{b_{I \times J} : I \times J \in \mathcal{R}\}$ we analyze the intersections $K \times L \cap \mathcal{E}_{I \times J}^*$ where $K \times L \in \mathcal{E}_{\tilde{I} \times J} \cup \mathcal{E}_{I \times \tilde{J}}$. Put

$$1 - \alpha_{I \times J} = \prod_{I_0 \times J_0 \leq I \times J} (1 - \delta_{I_0 \times J_0})^2.$$

We claim that

$$\frac{1}{2}(1 - \alpha_{I \times J})|K \times L| \leq |(K \times L) \cap \mathcal{E}_{I \times J}^*| \leq \frac{1}{2}|K \times L|, \quad (4.35a)$$

for all $K \times L \in \mathcal{E}_{\tilde{I} \times J}$ if $I \neq [0, 1]$, as well as

$$\frac{1}{2}(1 - \alpha_{I \times J})|K \times L| \leq |(K \times L) \cap \mathcal{E}_{I \times J}^*| \leq \frac{1}{2}|K \times L|, \quad (4.35b)$$

for all $K \times L \in \mathcal{E}_{I \times \tilde{J}}$ if $J \neq [0, 1]$.

Indeed, we only have to verify the left hand side estimates. First, let $K \times L \in \mathcal{E}_{\tilde{I} \times J}$. Observe that since $\mathcal{Y}_{\tilde{I} \times J} = \mathcal{Y}_{I \times J}$ and $|\mathcal{E}_{I \times J}^* \cap ([0, 1] \times L)| = \frac{1}{2} |\mathcal{X}_{I \times J}^*(L) \times L|$ for all $L \in \mathcal{Y}_{I \times J}$, we have

$$|(K \times L) \cap \mathcal{E}_{I \times J}^*| = \frac{1}{2} |(K \times L) \cap (\mathcal{X}_{I \times J}^*(L) \times L)|. \quad (4.36)$$

Obviously, by (4.29), the right hand side is larger than

$$\frac{1}{2} (1 - \delta_{I \times J}) |K \cap W_{I \times J}(L)| |L|. \quad (4.37)$$

We go back over the course by which we have come and see that

$$|K \cap W_{I \times J}(L)| \geq \prod_{\tilde{I} \times J \triangleleft I_0 \times J_0 \triangleleft I \times J} (1 - \delta_{I_0 \times J_0}) |K|. \quad (4.38)$$

Combining (4.36), with (4.37) and (4.38) yields (4.35a). Second, let $K \times L \in \mathcal{E}_{I \times \tilde{J}}$ and $J \neq [0, 1]$. By the definition of $\mathcal{E}_{I \times J}$ and (4.31) we have

$$|(K \times L) \cap \mathcal{E}_{I \times J}^*| = \sum_{L_0 \in \mathcal{Y}_{I \times J}} |(K \times L) \cap (\mathcal{X}_{I \times J}^*(L_0) \times L_0)|. \quad (4.39)$$

For each summand note the identity

$$|(K \times L) \cap (\mathcal{X}_{I \times J}^*(L_0) \times L_0)| = |K \cap \mathcal{X}_{I \times J}^*(L_0)| |L \cap L_0|. \quad (4.40)$$

As before, we have

$$|K \cap \mathcal{X}_{I \times J}^*(L_0)| \geq (1 - \delta_{I \times J}) |K \cap W_{I \times J}(L_0)| \quad (4.41)$$

and

$$|K \cap W_{I \times J}(L_0)| \geq \prod_{I \times \tilde{J} \triangleleft I_0 \times J_0 \triangleleft I \times J} (1 - \delta_{I_0 \times J_0}) |K|. \quad (4.42)$$

Next, we observe that by (4.40), (4.41) and (4.42), the sum in the right hand side of (4.39) is larger than

$$\left(\prod_{I \times \tilde{J} \triangleleft I_0 \times J_0 \triangleleft I \times J} (1 - \delta_{I_0 \times J_0}) \right) |K| \sum_{L_0 \in \mathcal{Y}_{I \times J}} |L \cap L_0|. \quad (4.43)$$

Taking into account that $J \subset \tilde{J}$, the Gamlen-Gaudet construction of Case 1 gives

$$\sum_{L_0 \in \mathcal{Y}_{I \times J}} |L \cap L_0| \geq \frac{1}{2} (1 - \delta_{[0, 1] \times J}) |L|. \quad (4.44)$$

Finally, combining (4.43) and (4.44) with (4.39) yields (4.35b).

Essential properties of our construction.

Output of the inductive step.

Having completed the construction of $\{\mathcal{E}_{I \times J} : I \times J \in \mathcal{R}_n\}$ we record the following crucial properties. First, (4.20) and (4.35) imply that for each $I_0 \times J_0, I_1 \times J_1 \in \mathcal{R}_n$ such that $I_0 \supset I_1, J_0 \supset J_1$ and $|I_0 \times J_0| = 2 |I_1 \times J_1|$ we have

$$\frac{1}{2} \prod_{I \times J \triangleleft I_1 \times J_1} (1 - \delta_{I \times J})^2 |K \times L| \leq |(K \times L) \cap \mathcal{E}_{I_1 \times J_1}^*| \leq \frac{1}{2} |K \times L|, \quad (4.45)$$

for all $K \times L \in \mathcal{E}_{I_0 \times J_0}$. Second, (4.11), (4.12), (4.18) and (4.19) as well as (4.24), (4.25), (4.32) and (4.33) imply

$$\sum_{j=1}^{i-1} |\langle T^* b_j, h_{K \times L} \rangle| + |\langle T b_j, h_{K \times L} \rangle| \leq \frac{(i-1) \tau_i}{\beta_i} |K \times L|, \quad (4.46)$$

for all $i \in \mathbb{N}$ and $K \times L \in \mathcal{E}_i$. Recall that $\mathcal{E}_i = \mathcal{E}_{I \times J}$ provided $i = \mathcal{O}_\triangleleft(I \times J)$.

Bi-tree property.

The collection $\{\mathcal{E}_{I \times J}^* : I \times J \in \mathcal{R}_n\}$ forms a bi-tree, see (2.3). The bi-tree constant is determined by the local product structure (4.48) verified below. In particular

$$\frac{1}{2}|I \times J| \leq |\mathcal{E}_{I \times J}^*| \leq |I \times J|. \quad (4.47)$$

The local product structure of $\mathcal{E}_{I \times J}$.

Here, we exploit our choice of the constants $\delta_{I \times J}$, see (4.10). We carry over (4.45) to each pair of nested dyadic rectangles. Let $I_0 \times J_0, I_1 \times J_1 \in \mathcal{R}$ such that $I_0 = \pi^i(I_1)$ and $J_0 = \pi^j(J_1)$ for some $i, j \in \mathbb{N}_0$. Then, iterating (4.45) yields

$$\frac{1}{2}|K \times L| \leq 2^{i+j}|(K \times L) \cap \mathcal{E}_{I_1 \times J_1}^*| \leq |K \times L|, \quad (4.48)$$

for all $K \times L \in \mathcal{E}_{I_0 \times J_0}$. Our construction with its inherent complications permits us now verify the crucial estimate (4.48). We present only the proof for the lower estimate since the verification of the upper estimate follows the same line of reasoning. Let $I_0 \times J_0$ and $I_1 \times J_1$ be a nested pair of dyadic rectangles as specified above. We now define a path $p(I_0 \times J_0, I_1 \times J_1)$ of nested rectangles $I^{(m)} \times J^{(m)}$ connecting $I_1 \times J_1$ to $I_0 \times J_0$ as follows. We define $I^{(0)} = I_1$, $I^{(i+j)} = I_0$ and $J^{(0)} = J_1$, $J^{(i+j)} = J_0$ as well as

$$\begin{aligned} I^{(m+1)} &= \tilde{I}^{(m)} \quad \text{and} \quad J^{(m+1)} = J^{(m)}, & \text{if } 0 \leq m \leq i-1, \\ I^{(m+1)} &= I^{(m)} \quad \text{and} \quad J^{(m+1)} = \tilde{J}^{(m)}, & \text{if } i \leq m \leq i+j-1. \end{aligned}$$

Iterating the local property (4.45) along the path $p = p(I_0 \times J_0, I_1 \times J_1)$ we obtain

$$|(K \times L) \cap \mathcal{E}_{I_1 \times J_1}^*| \geq 2^{-(i+j)} \prod_{I \times J \in p} (1 - \alpha_{I \times J}) |K \times L|,$$

where we put

$$1 - \alpha_{I \times J} = \prod_{k \leq \mathcal{O}_\triangleleft(I \times J)} (1 - \delta_k)^2.$$

Since the length of the path p is at most $2n$, we obtain

$$|(K \times L) \cap \mathcal{E}_{I_1 \times J_1}^*| \geq 2^{-(i+j)} |K \times L| \left(1 - 4n \sum_{k=1}^{\infty} \delta_k\right).$$

As we specified $\delta_k = 2^{-k}/(8n)$ in (4.10) we see that (4.48) holds.

The boundedness of the orthogonal projection Q .

The collections of dyadic rectangles $\mathcal{E}_{I \times J}$ gives rise to the block basis

$$b_{I \times J} = \sum_{K \times L \in \mathcal{E}_{I \times J}} h_{K \times L}$$

and the orthogonal projection

$$Q(f) = \sum_{I \times J \in \mathcal{R}_n} \langle f, \frac{b_{I \times J}}{\|b_{I \times J}\|_2} \rangle \frac{b_{I \times J}}{\|b_{I \times J}\|_2}.$$

Feeding the estimate (4.48) into Theorem 2.2 we obtain that

$$\|Q : H^1(\delta^2) \rightarrow H^1(\delta^2)\| \leq C_2,$$

for some universal constant $C_2 > 0$.

The basis $\{b_i\}$ are almost eigenvectors for T .

We show that we have

$$Tb_i = \frac{|\langle Tb_i, b_i \rangle|}{\|b_i\|_2^2} b_i + \text{tiny error}.$$

To be more precise, we claim that

$$\sum_{j:j \neq i} |\langle Tb_j, b_i \rangle| \leq \varepsilon_i \|b_i\|_2^2, \quad \text{for all } i. \quad (4.49)$$

We begin the proof of (4.49) by summing estimate (4.46) over all $K \times L \in \mathcal{E}_i$ to obtain

$$\sum_{j=1}^{i-1} |\langle T^* b_j, b_i \rangle| + |\langle Tb_j, b_i \rangle| \leq \frac{(i-1)\tau_i}{\beta_i} \|b_i\|_2^2. \quad (4.50)$$

Reversing the roles of i and j in (4.50) gives

$$|\langle Tb_j, b_i \rangle| = |\langle b_j, T^* b_i \rangle| \leq \frac{(j-1)\tau_j}{\beta_j} \|b_j\|_2^2, \quad j \geq i+1. \quad (4.51)$$

Taking the sum in (4.51) and adding (4.50) we get

$$\sum_{j:j \neq i} |\langle Tb_j, b_i \rangle| \leq 2 \sum_{j \geq i} \frac{(j-1)\tau_j}{\beta_j} \|b_j\|_2^2. \quad (4.52)$$

Now, recall that in (4.15) and (4.27) we defined the constants τ_j by

$$\tau_j = \frac{2^{-j}}{4(j-1)} \beta_j \min_{k \leq j} \|b_k\|_2^2 \varepsilon_k \quad (4.53)$$

Finally, plugging (4.53) into (4.47) yields

$$\sum_{j:j \neq i} |\langle Tb_j, b_i \rangle| \leq 2^{-i} \varepsilon_i \|b_i\|_2^2,$$

which is certainly smaller than estimate (4.49).

5. LOCALIZATION IN BI-PARAMETER BMO

In this section we give the proof of our main theorem 1.1 restated below.

Theorem (Main theorem 1.1). *For any operator*

$$T : BMO(\delta^2) \rightarrow BMO(\delta^2)$$

the identity on $BMO(\delta^2)$ factors through $H = T$ or $H = \text{Id} - T$, that is

$$\begin{array}{ccc} BMO(\delta^2) & \xrightarrow{\text{Id}} & BMO(\delta^2) \\ E \downarrow & & \uparrow P \\ BMO(\delta^2) & \xrightarrow{H} & BMO(\delta^2) \end{array} \quad \|E\| \|P\| \leq C. \quad (5.1)$$

The structure of the proof given below follows the general localization method introduced by Bourgain [4] to treat factorization problems. We first list the basic steps of the argument:

- (i) We exploit Wojtaszczyk's isomorphism asserting that

$$BMO(\delta^2) \sim \left(\sum_n BMO_n(\delta^2) \right)_\infty.$$

- (ii) We reduce the factorization problem to the case where the operator T is a diagonal operator on $\left(\sum_n BMO_n(\delta^2) \right)_\infty$.

(iii) We invoke our finite dimensional factorization Theorem 3.1 to infer that in fact Theorem 1.1 holds true for diagonal operators.

We say an operator $D : (\sum_n BMO_n(\delta^2))_\infty \rightarrow (\sum_n BMO_n(\delta^2))_\infty$ is a diagonal operator if there exists a sequence of operators $A_n : BMO_n(\delta^2) \rightarrow BMO_n(\delta^2)$ such that

$$D(f_1, f_2, \dots, f_n, \dots) = (A_1 f_1, A_2 f_2, \dots, A_n f_n, \dots).$$

The following theorem provides the reduction to diagonal operators.

Theorem 5.1. *For any linear operator $T : (\sum_n BMO_n(\delta^2))_\infty \rightarrow (\sum_n BMO_n(\delta^2))_\infty$ there exists a diagonal operator*

$$D : (\sum_n BMO_n(\delta^2))_\infty \rightarrow (\sum_n BMO_n(\delta^2))_\infty$$

and bounded linear operators

$$R, E : (\sum_n BMO_n(\delta^2))_\infty \rightarrow (\sum_n BMO_n(\delta^2))_\infty$$

such that

$$D = RTE \quad \text{and} \quad \text{Id} - D = R(\text{Id} - T)E. \quad (5.2)$$

We remark that (5.2) implies $RE = \text{Id}$.

The proof of Theorem 5.1 relies on the repeated application of the following theorem which is a simplified variant of Theorem 3.2.

Theorem 5.2. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$, then there exists an $N = N(n, \varepsilon)$ so that the following holds. For any n -dimensional subspace $F \subset BMO_N(\delta^2)$, there exists a block-basis $\{b_{I \times J}\}$ in $BMO_N(\delta^2)$ satisfying the following conditions.*

(i) *The map $S : BMO_n(\delta^2) \rightarrow BMO_N(\delta^2)$ defined as the linear extension of $h_{I \times J} \mapsto b_{I \times J}$ satisfies*

$$\frac{1}{C} \|f\| \leq \|Sf\| \leq C \|f\|,$$

with universal constant C .

(ii) *The orthogonal projection $Q : BMO_N(\delta^2) \rightarrow BMO_N(\delta^2)$ given by*

$$Q(f) = \sum_{I \times J \in \mathcal{A}_n} \langle f, \frac{b_{I \times J}}{\|b_{I \times J}\|_2} \rangle \frac{b_{I \times J}}{\|b_{I \times J}\|_2}$$

is bounded by

$$\|Qf\|_{BMO_N(\delta^2)} \leq C \|f\|_{BMO_N(\delta^2)}, \quad f \in BMO_N(\delta^2),$$

for some universal constant C and almost annihilates the space F ,

$$\|Qf\|_{BMO_N(\delta^2)} \leq \varepsilon \|f\|_{BMO_N(\delta^2)}, \quad f \in F. \quad (5.3)$$

Proof. The proof of Theorem 5.2 is a repetition of the almost-diagonalization argument in the proof of Theorem 3.2, where condition (5.3) is simpler to realize than (3.2). The situation is analogous to the one parameter case treated in [17, 290–291]. \square

Proof of Theorem 5.1. The proof of Theorem 5.1 is quantitative and finite dimensional in nature. The estimates pertaining specifically to bi-parameter BMO are provided by Theorem 5.2. The reduction procedure itself is analogous to the corresponding localization theorems in [2, 4, 17, 20].

Let $\varepsilon > 0$ and $\varepsilon_n = 4^{-n-3}\varepsilon$. Subsequently, we write $N = N(n) = N(n, \varepsilon_n)$ as specified in Theorem 5.2. We further abbreviate

$$X_n = BMO_n(\delta^2) \quad \text{and} \quad X = (\sum_n X_n)_\infty.$$

Let $p_j : X \rightarrow X_j$ denote the projection onto the j -th coordinate. Given a subset Λ of \mathbb{N} we define $P_\Lambda : X \rightarrow X$ by

$$p_j P_\Lambda x = \begin{cases} x_j, & \text{if } j \in \Lambda \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } x = (x_n)_n \in X, j \in \mathbb{N}.$$

We will now inductively define an increasing sequence of integers $M(n)$, a decreasing sequence of infinite subsets Λ_n of \mathbb{N} , subspaces F_n of $X_{M(n)}$ (see (5.4) below), projections $Q_n : X_{M(n)} \rightarrow X_{M(n)}$ and isomorphisms $S_n : X_n \rightarrow Q(X_{M(n)})$ such that

- (i) $\|Q_n\| \leq C$ and $\|S_n\| \|S_n^{-1}\| \leq C$,
- (ii) $\|Q_n(x)\|_{X_{M(n)}} \leq \varepsilon_n \|x\|_{X_{M(n)}}$ for all $x \in F_n$,
- (iii) $M(n) \in \Lambda_n$ and $\min \Lambda_n > M(n-1)$,
- (iv) $\|p_{M(n-1)} T P_{\Lambda_n}\| \leq \varepsilon_n$.

We begin the construction by defining $M(1) = 1$, $\Lambda_1 = \mathbb{N}$, $Q_1 = \text{Id}$ and $F_1 = \{0\}$. Assume we have completed our construction for all $1 \leq j \leq n-1$. We will now choose an infinite subset Λ_n of Λ_{n-1} such that (iii) and (iv) are satisfied. Since $X_{M(n-1)}$ is finite dimensional it suffices to show that for every $\varphi \in X^*$ there exists an infinite subset Λ_n of Λ_{n-1} such that

$$\|\varphi P_{\Lambda_n}\| \leq \varepsilon_n.$$

To this end let $\varphi \in X^*$ and $\Gamma = \{k \in \Lambda_{n-1} : k > M(n-1)\}$. Assume that for each infinite subset Λ of Γ we have that $\|\varphi P_\Lambda\| > \varepsilon_n$. Partition the infinite set Γ into m disjoint infinite sets $\Gamma_1, \dots, \Gamma_m$ and choose $x_1, \dots, x_m \in X$ with $\|x_j\| = 1$ such that $\varphi P_{\Gamma_j} x_j > \varepsilon_n$. Observe that the disjointness of the Γ_j implies that $\|\sum_{j=1}^m P_{\Gamma_j} x_j\| \leq 1$, thus

$$m\varepsilon_n < \sum_{j=1}^m \varphi P_{\Gamma_j} x_j \leq \|\varphi\|.$$

This gives a contradiction for sufficiently large m , showing (iii) and (iv).

Let the projection $Q^{(n-1)} : X \rightarrow X$ be defined by

$$p_j Q^{(n-1)} x = \begin{cases} Q_k x_k, & \text{if } j = M(k) \text{ and } j \leq M(n-1) \\ 0, & \text{otherwise} \end{cases}$$

for all $x = (x_k)_k \in X$ and $j \in \mathbb{N}$. Then define the subspace $W_n = TQ^{(n-1)}(X)$ and choose $M(n) = \min\{k \in \Lambda_n : k \geq N(\dim W_n, \varepsilon_n)\}$, where $N = N(\dim W_n, \varepsilon_n)$ is the constant appearing in Theorem 5.2. We next specify a subspace F_n by putting

$$F_n = p_{M(n)} W_n. \quad (5.4)$$

Theorem 5.2 asserts that there exists a projection Q_n and an isomorphism $S_n : X_n \rightarrow Q_n(X_{M(n)})$ such that (i) and (ii) are satisfied.

We will now define the maps $I, Q : X \rightarrow X$ by

$$p_j Ix = \begin{cases} S_n x_n, & \text{if } j = M(n) \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad p_j Qx = \begin{cases} Q_n x_n, & \text{if } j = M(n) \\ 0, & \text{otherwise} \end{cases}$$

for all $x = (x_n)_n \in X$ and $j \in \mathbb{N}$. Define $J : Q(X) \rightarrow X$ by

$$Jy = (S_n^{-1} y_{M(n)})_n \quad \text{for all } y = (y_n)_n \in Q(X).$$

Note that $JQI = \text{Id}$ and that therefore

$$\widehat{T} = JQTI \quad (5.5)$$

satisfies

$$\text{Id} - \widehat{T} = JQ(\text{Id} - T)I \quad (5.6)$$

and moreover \widehat{T} is a small perturbation of a diagonal operator. Indeed, define $D : X \rightarrow X$ by $D = (p_n \widehat{T} p_n)_n$ and observe that D is a bounded diagonal operator for which

$$\|\widehat{T} - D\| < \varepsilon, \quad (5.7)$$

since we chose $\varepsilon_n = 4^{-n-3}\varepsilon$. This is a consequence of conditions (i) to (iv). A standard perturbation argument shows finally the existence of the operators

$$R, E : X \rightarrow X$$

such that

$$D = RTE \quad \text{and} \quad \text{Id} - D = R(\text{Id} - T)E. \quad \square$$

In Theorem 5.1 we provided the reduction of the general factorization theorem 1.1 to the case of diagonal operators. We now turn to the remaining last step: we show that the factorization theorem holds true for diagonal operators.

Theorem 5.3. *Let D be a diagonal operator on $(\sum_n BMO_n(\delta^2))_\infty$. Then the identity factors through $H = D$ or $H = \text{Id} - D$, that is*

$$\begin{array}{ccc} (\sum_n BMO_n(\delta^2))_\infty & \xrightarrow{\text{Id}} & (\sum_n BMO_n(\delta^2))_\infty \\ E \downarrow & & \uparrow P \\ (\sum_n BMO_n(\delta^2))_\infty & \xrightarrow{H} & (\sum_n BMO_n(\delta^2))_\infty \end{array} \quad \|E\|\|P\| \leq C,$$

where C is a universal constant.

Proof. Let $A_n : BMO_n(\delta^2) \rightarrow BMO_n(\delta^2)$ be the linear map defining the diagonal operator D , that is

$$D(f_1, f_2, \dots, f_n, \dots) = (A_1 f_1, A_2 f_2, \dots, A_n f_n, \dots).$$

By Theorem 3.1 the identity on $BMO_n(\delta^2)$ factors through $H_n = A_{N(n)}$ or $H_n = \text{Id} - A_{N(n)}$, that is

$$\begin{array}{ccc} BMO_n(\delta^2) & \xrightarrow{\text{Id}} & BMO_n(\delta^2) \\ E_n \downarrow & & \uparrow P_n \\ BMO_n(\delta^2) & \xrightarrow{H_n} & BMO_n(\delta^2) \end{array} \quad \|E_n\|\|P_n\| \leq C,$$

for some universal constant $C > 0$. If there exists an infinite sequence $\{k(n)\}$ so that $H_{k(n)} = A_{N(k(n))}$, then the identity on $(\sum_n BMO_n(\delta^2))_\infty$ factors through D . If $H_{k(n)} = \text{Id} - A_{N(k(n))}$, then the identity factors through $\text{Id} - D$. \square

We now combine theorems 5.1 and 5.3 and derive Theorem 1.1.

Proof of Theorem 1.1. By Wojtaszczyk's isomorphism, see [21], the Banach space $BMO(\delta^2)$ is isomorphic to the infinite sum of its finite dimensional building blocks $(\sum_n BMO_n(\delta^2))_\infty$. Hence, in Theorem 1.1 we replace operators on $BMO(\delta^2)$ by operators on $(\sum_n BMO_n(\delta^2))_\infty$. Moreover, by Theorem 5.1, it suffices to consider only **diagonal** operators on $(\sum_n BMO_n(\delta^2))_\infty$. In Theorem 5.3 we proved that for any diagonal operator D on $(\sum_n BMO_n(\delta^2))_\infty$ the identity factors through $H = D$ or $H = \text{Id} - D$, that is

$$\begin{array}{ccc} (\sum_n BMO_n(\delta^2))_\infty & \xrightarrow{\text{Id}} & (\sum_n BMO_n(\delta^2))_\infty \\ E \downarrow & & \uparrow P \\ (\sum_n BMO_n(\delta^2))_\infty & \xrightarrow{H} & (\sum_n BMO_n(\delta^2))_\infty \end{array} \quad \|E\|\|P\| \leq C.$$

for some universal constant $C > 0$. □

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